

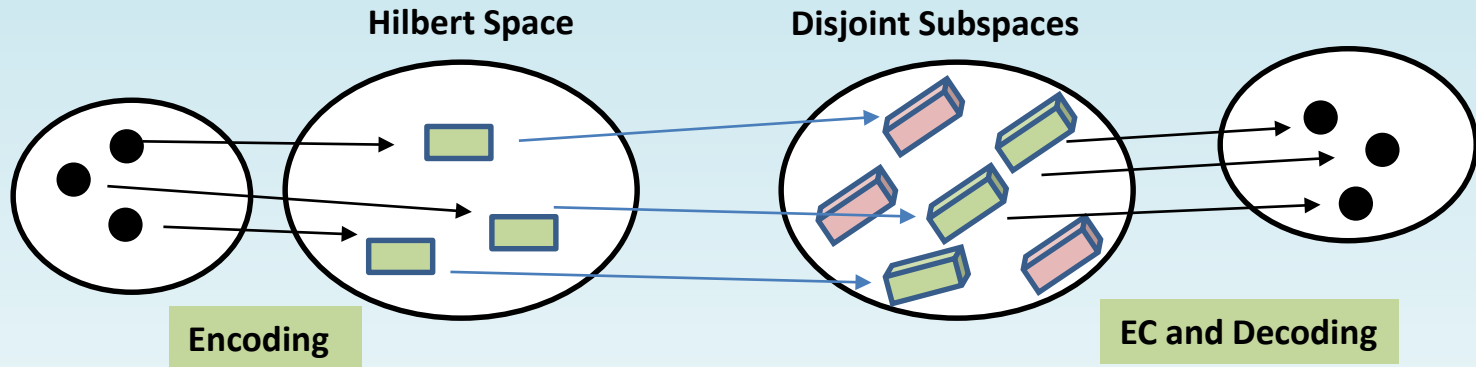
Fault-Tolerant Quantum Error Correction

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OUTLINE

- ❖ Quantum codes
- ❖ What is fault-tolerance?
- ❖ Fault-tolerant quantum error correction methods
- ❖ Universal Fault tolerant quantum computation

QUANTUM CODES

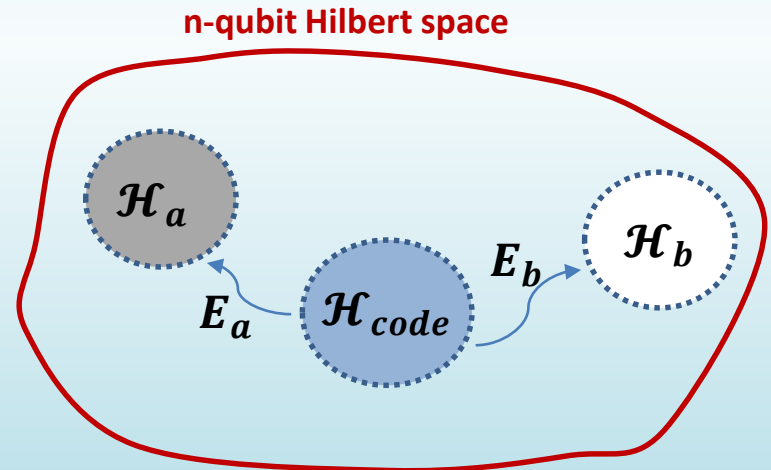


$$\begin{aligned} |0\rangle &\rightarrow |000\rangle \\ |1\rangle &\rightarrow |111\rangle \end{aligned}$$

$$\mathcal{E} = \{E_a\}$$

Basis for n – qubit errors :

$$E = O_1 \otimes O_2 \otimes \dots \otimes O_n, O_i \in \{\mathbb{I}, X, Y, Z\}$$

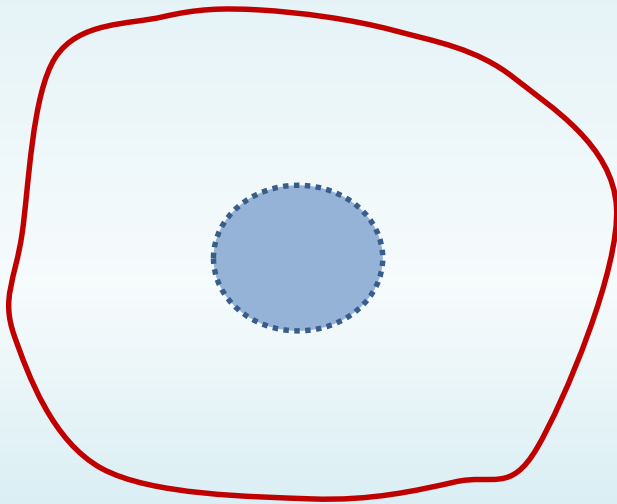


QUANTUM CODES

Code parameters : $[[n, k, d]]$ $t = \left\lfloor \frac{d-1}{2} \right\rfloor$

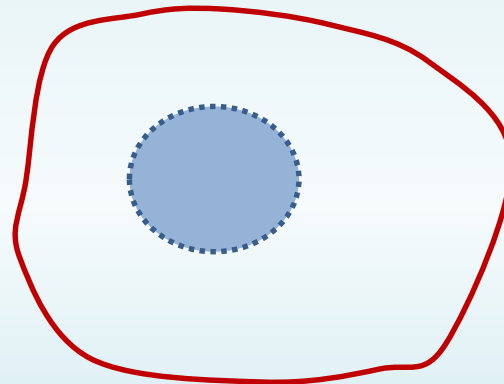
Distance, rate (k/n), useful for FTQC

9-qubit Hilbert space



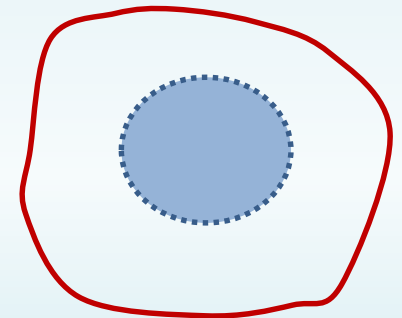
$[[9, 1, 3]]$

7-qubit Hilbert space



$[[7, 1, 3]]$

5-qubit Hilbert space

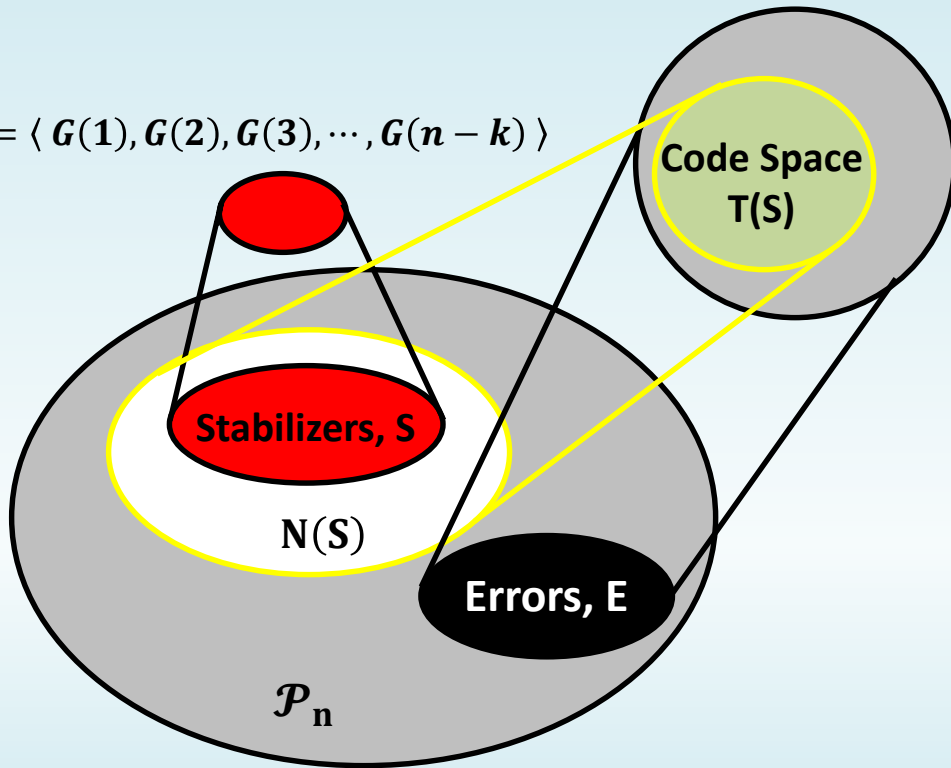


$[[5, 1, 3]]$

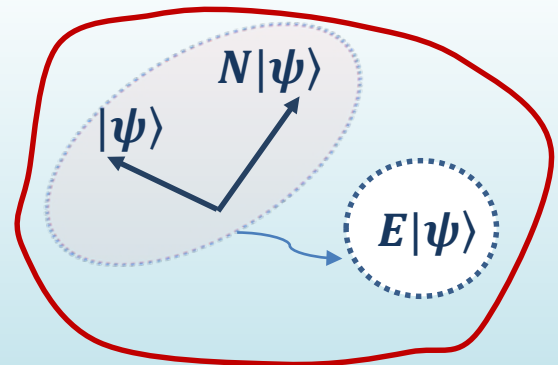
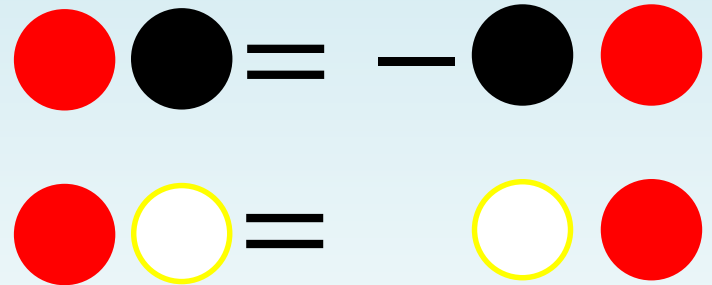
STABILIZER CODES

$$T(S) = \{ s|\psi\rangle = |\psi\rangle \}$$

$$G = \langle G(1), G(2), G(3), \dots, G(n-k) \rangle$$



$$\mathcal{P}_n = c \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \dots \otimes \mathcal{O}_n$$



STABILIZER CODES

n- qubit stabilizer code: $[[n, k, d]]$

$$G = \langle G(1), G(2), G(3), \dots, G(n - k) \rangle$$

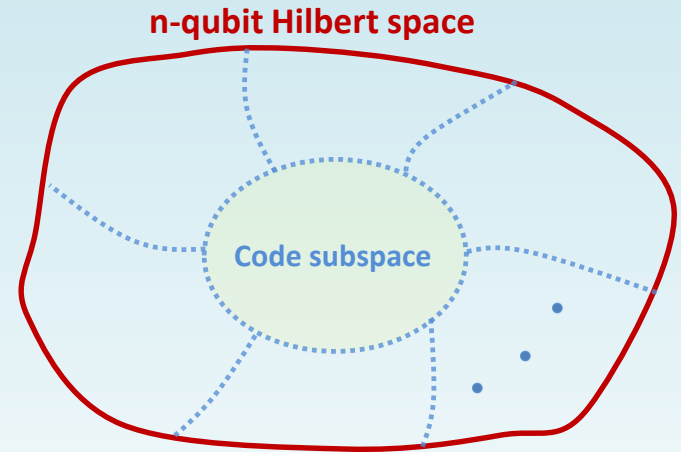
$$G^2(i) = \mathbb{I}$$

each subspace $\xrightarrow{\text{LABEL}}$ $e = (e_1, e_2, \dots, e_{n-k})$

$$G(i) \xrightarrow{\text{eigenvalue}} (-1)^{e_i}$$

$e \equiv$ subspace syndrome

2^n -D Hilbert space $\xrightarrow{\text{decomposes into } 2^{n-k}}$ disjoint eigenspaces, each of $D = 2^k$

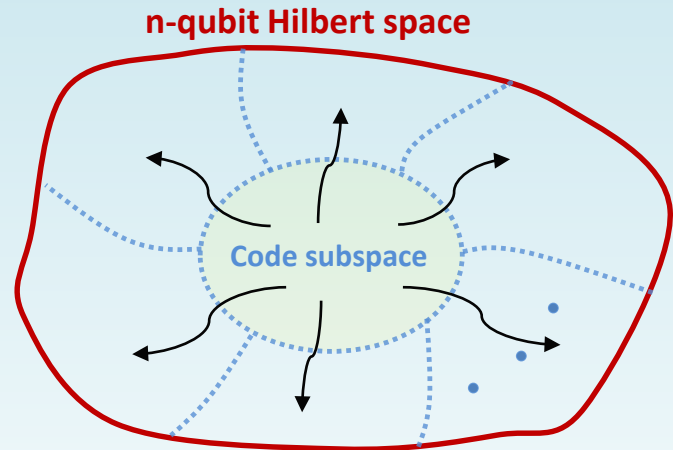


HOW TO DETECT ERRORS?

code space syndrome : $e = (00 \dots 0)$

action of P_a on code space $\xrightarrow{\text{change syndrome}}$ $e = a\Lambda G^T$

$$XIIYZYI \rightarrow \underbrace{(XIXIXI)}_{(101010)} \underbrace{(IIZZZI)}_{(001110)}$$



$$P_a = \otimes X^{a_i} \otimes Z^{a_{n+i}} \rightarrow a := (x_a : z_a)$$

$$x_a \cdot z_b + z_a \cdot x_b = 0$$

$$G(i) = \otimes X^{b_i} \otimes Z^{b_{n+i}} \rightarrow g(i) := (x_b : z_b)$$

$$\Lambda = \begin{pmatrix} \mathbf{0}_n & \mathbb{I}_n \\ \mathbb{I}_n & \mathbf{0}_n \end{pmatrix}$$

$$a\Lambda g(i)^T = 0$$

$$G \equiv \begin{pmatrix} g(1) \\ g(2) \\ \vdots \\ g(n-k) \end{pmatrix}$$

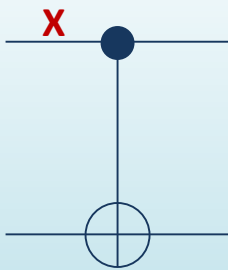
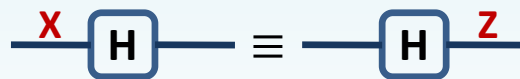
WHAT IS FAULT-TOLERANCE?

ERROR PROPAGATION :

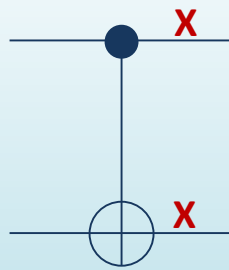


$$UE|\psi\rangle = UE(U^\dagger U)|\psi\rangle = (UEU^\dagger)|\psi'\rangle$$

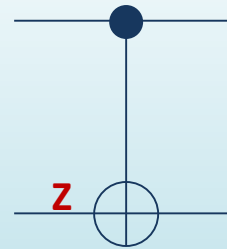
$$E \rightarrow UEU^\dagger$$



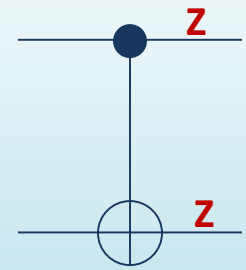
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FAULT-TOLERANCE

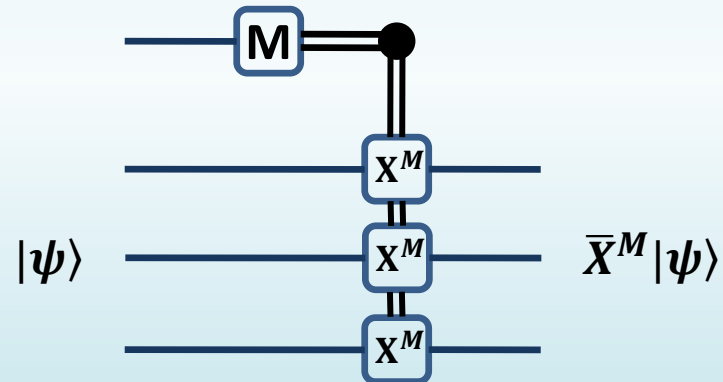
$$\mathbf{X} \otimes \mathbb{I} \rightarrow \mathbf{X} \otimes \mathbf{X}$$

$$\mathbf{Z} \otimes \mathbb{I} \rightarrow \mathbf{Z} \otimes \mathbb{I}$$

$$\mathbb{I} \otimes \mathbf{X} \rightarrow \mathbb{I} \otimes \mathbf{X}$$

$$\mathbb{I} \otimes \mathbf{Z} \rightarrow \mathbf{Z} \otimes \mathbf{Z}$$

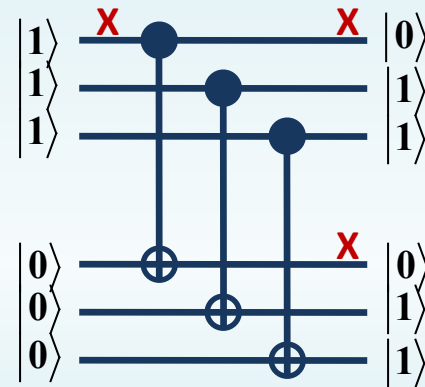
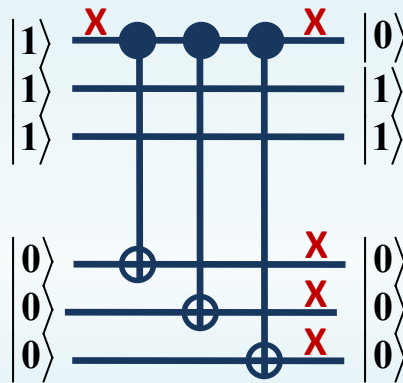
Faulty measurements can cause errors to cascade



How to avoid cascading?

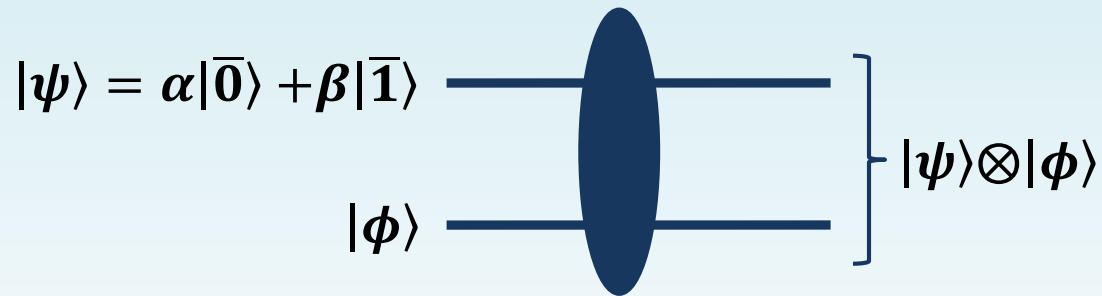
Transversal Gates

$$|111\rangle|000\rangle \rightarrow |111\rangle|111\rangle$$



FAULT-TOLERANT ERROR CORRECTION METHODS

Error correction = Syndrome measurement(S. M.) + Recovery step

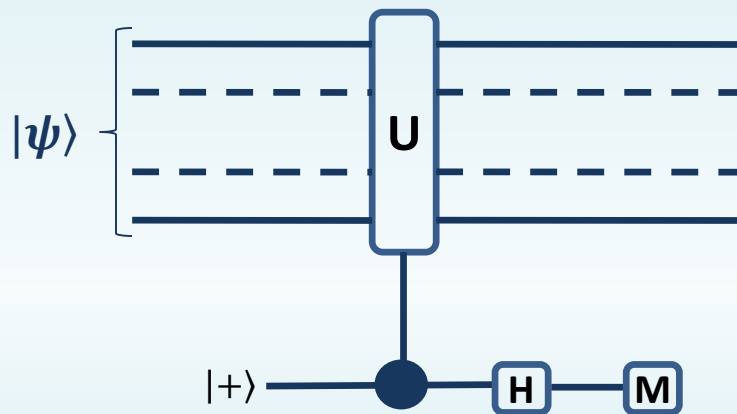


- ✓ **Shor type (S.M. using Cat states)**
- ✓ **Steane type (S.M. using encoded blocks)**
- ✓ **Knill type (S.M via Logical Teleportation)**

SHOR TYPE ERROR CORRECTION

Syndrome measurement using Cat states :

Phase kickback trick :



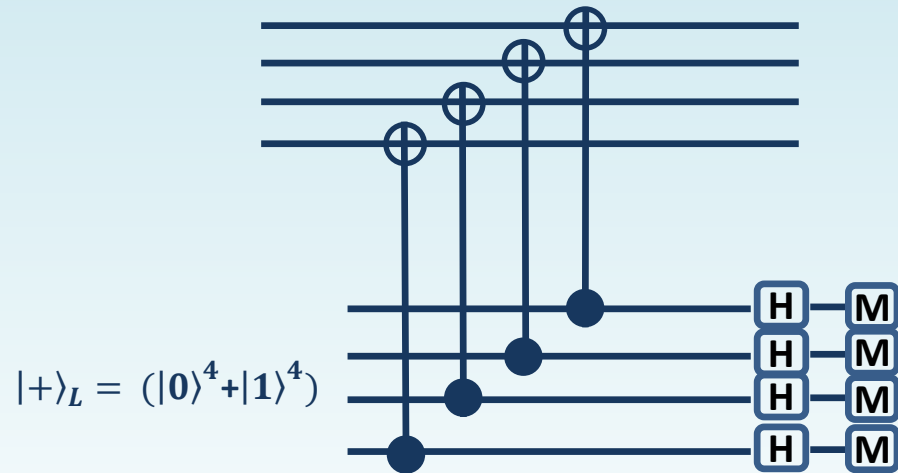
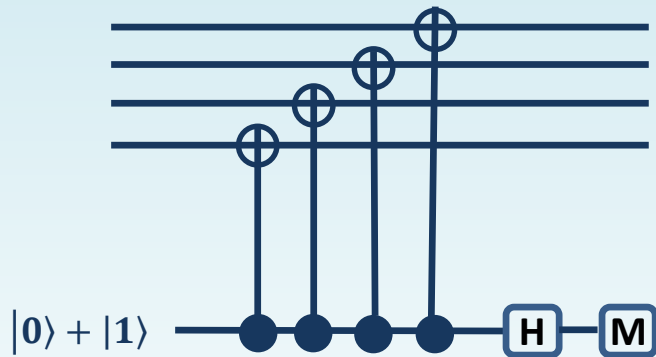
$$|+\rangle|\psi\rangle \xrightarrow{C-U} |0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle$$

$$= (|0\rangle \pm |1\rangle)|\psi\rangle \xrightarrow{H^{\otimes 1}} |0 \text{ or } 1\rangle \otimes |\psi\rangle$$

$$G = \langle G(1), G(2), G(3), \dots, G(n - k) \rangle$$

SHOR TYPE ERROR CORRECTION

$$U = X^{\otimes 4}$$



$$H^{\otimes n}(|0\rangle^4 + |1\rangle^4) = \sum_{w(x)=\text{even}} |x\rangle$$

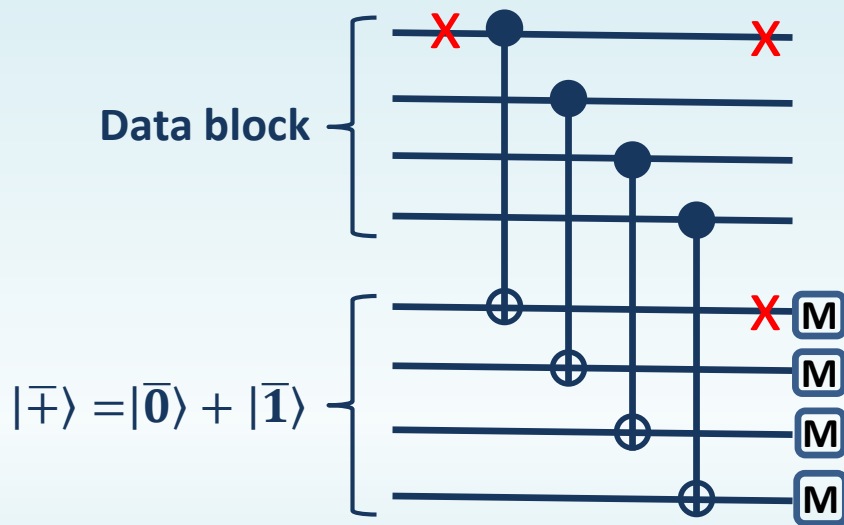
$$H^{\otimes n}(|0\rangle^4 - |1\rangle^4) = \sum_{w(x)=\text{odd}} |x\rangle$$

after the complete syndrome e is measured $\xrightarrow{e = a\Lambda G^T}$ apply to correct the error

STEANE TYPE ERROR CORRECTION

Ancillas → encoded using the same CSS QECC

To measure Z type generators :



$$Z(b) = \bigotimes_{i=1}^n Z^{b_i} \xrightarrow{\text{binary vector}} \mathbf{z}_b = (b_1, b_2, \dots, b_n)$$

Measurement result $\rightarrow \mathbf{x}_a = (x_1, x_2, \dots, x_n)$

Eigenvalue of $Z(b) \rightarrow \mathbf{x}_a \cdot \mathbf{z}_b$

STEANE TYPE ERROR CORRECTION

7 – qubit code: $[[7, 1, 3]]$

$$G(1) = IIIIXXXX$$

$$G(2) = XIXIXIX$$

$$G(3) = IXXIIXX$$

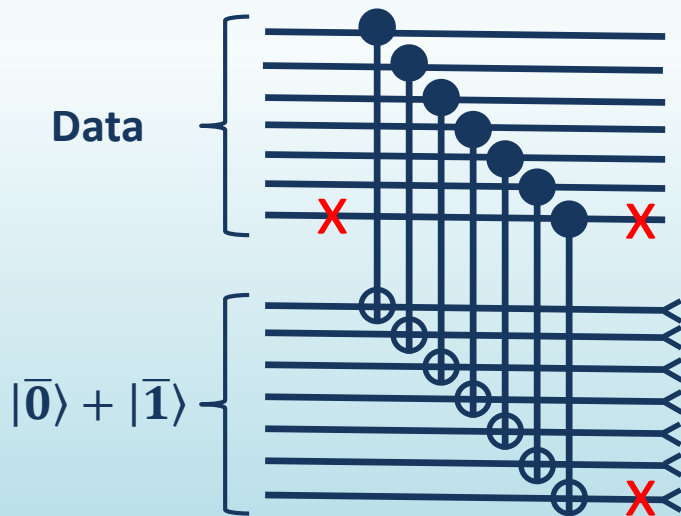
$$G(4) = IIIIZZZZ$$

$$G(5) = ZIZIZIZ$$

$$G(6) = IZZIIZZ$$

$$|\bar{0}\rangle = |0000000\rangle + |0001111\rangle + |0111100\rangle + |1100110\rangle \\ + |0110011\rangle + |1010101\rangle + |1011010\rangle + |1101001\rangle$$

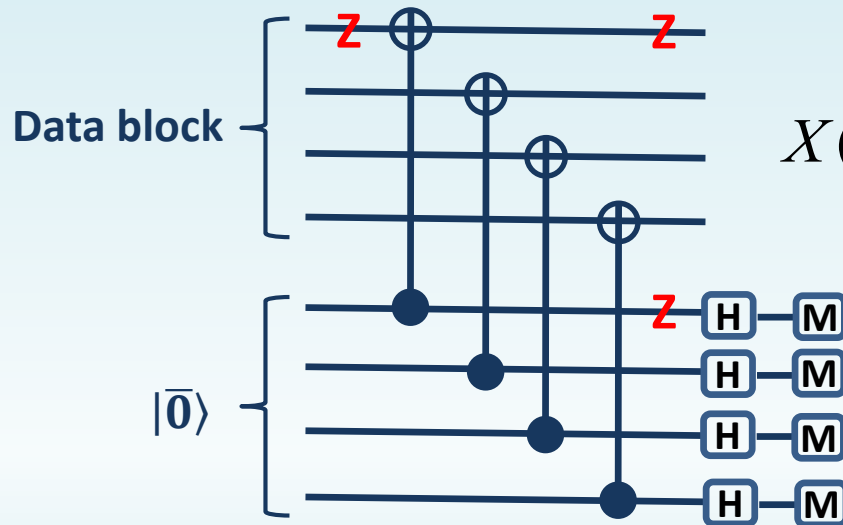
$$|\bar{1}\rangle = |1111111\rangle + |1110000\rangle + |1000011\rangle + |0011001\rangle \\ + |1001100\rangle + |0101010\rangle + |0100101\rangle + |0010110\rangle$$



$$G(4) \xleftrightarrow{\text{Binary vector}} z_b := (0001111)$$

STEANE TYPE ERROR CORRECTION

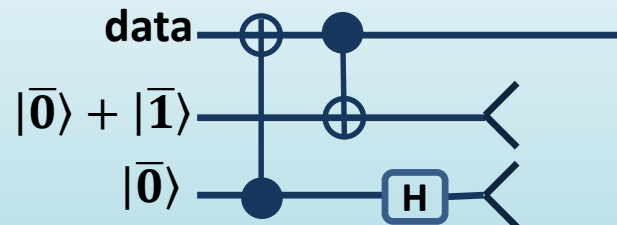
To measure X type generators :



$$X(b) = \bigotimes_{i=1}^n X^{b_i} \xrightarrow{\text{binary vector}} x_b = (b_1, b_2, \dots, b_n)$$

Measurement result $\rightarrow z_a = (z_1, z_2, \dots, z_n)$

Eigenvalue of $X(b) \rightarrow z_a \cdot x_b$



Universal Fault tolerant quantum computation

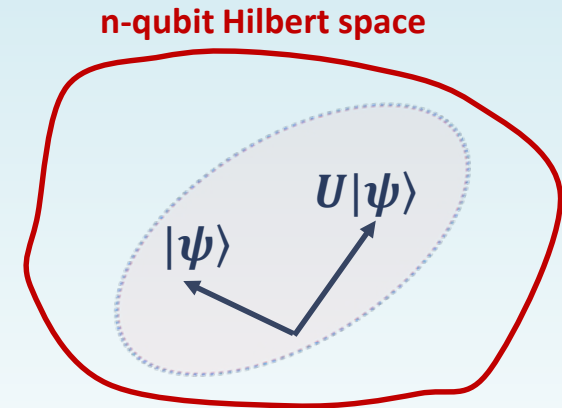
U (arbitrary unitary) \rightarrow How does this affect code stabilizers?

$$S|\psi\rangle = |\psi\rangle$$

$$US|\psi\rangle = U|\psi\rangle$$

$$(USU^\dagger)U|\psi\rangle = U|\psi\rangle$$

$$USU^\dagger \rightarrow S$$



Clifford operators :

$$\mathcal{C}_2 = \{U | UQU^\dagger \in \mathcal{P}_n\} \rightarrow \text{generators : } \{H, P, CNOT\}$$

$$H : \begin{cases} X \rightarrow Z \\ Z \rightarrow X \end{cases}$$

$$P : \begin{cases} X \rightarrow iY \\ Z \rightarrow Z \end{cases}$$

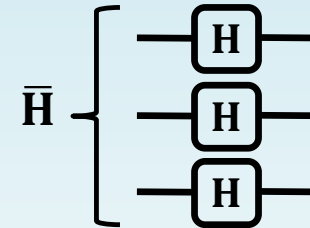
Universal gate sets for QC : $\{H, P, CNOT, T_{\pi/8}\}$

F-T IMPLEMENTATION OF CLIFFORD GROUP

Fault-Tolerant Hadamard Gate :

H: switches X and Z → Symmetry → **self duality**

$$\begin{aligned}
 G(1) &= \text{IIIXXXX} & G(4) &= \text{IIIZZZZ} \\
 G(2) &= \text{XIXIXIX} & G(5) &= \text{ZIZIZIZ} \\
 G(3) &= \text{IXXIIIXX} & G(6) &= \text{IZZIIZZ} \\
 \bar{X} &= \text{XXXIIIII} \\
 \bar{Z} &= \text{ZZZIIIII}
 \end{aligned}$$

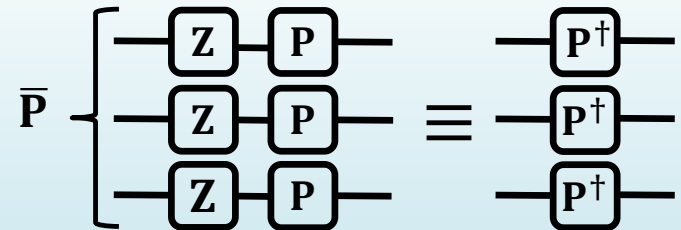


Fault-Tolerant Phase Gate :

$$P : \begin{cases} X \rightarrow iY \\ Z \rightarrow Z \end{cases}$$

take care about overall phase → **doubly-even codes**

\bar{P} must be a logical phase gate!!



$$PZXZP^\dagger = -iY$$

F-T IMPLEMENTATION OF CLIFFORD GROUP

Fault-Tolerant CNOT Gate :

CSS codes $\xrightarrow{\text{are the only codes}}$ for which bitwise CNOT is a valid F-T operation

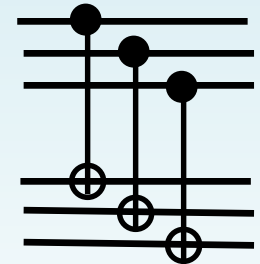
a generic stabilizer : $S = MN$

$$MN \otimes \mathbb{I} \xrightarrow{CNOT} MN \otimes M$$

$$S \otimes \mathbb{I} \ \& \ \mathbb{I} \otimes S$$

$$S \otimes \mathbb{I} \rightarrow S \otimes S \quad \text{if } S \text{ be } X\text{-generator}$$

$$\mathbb{I} \otimes S \rightarrow S \otimes S \quad \text{if } S \text{ be } Z\text{-generator}$$



$$\bar{X} \otimes \mathbb{I} \rightarrow \bar{X} \otimes \bar{X}$$

$$\bar{Z} \otimes \mathbb{I} \rightarrow \bar{Z} \otimes \mathbb{I}$$

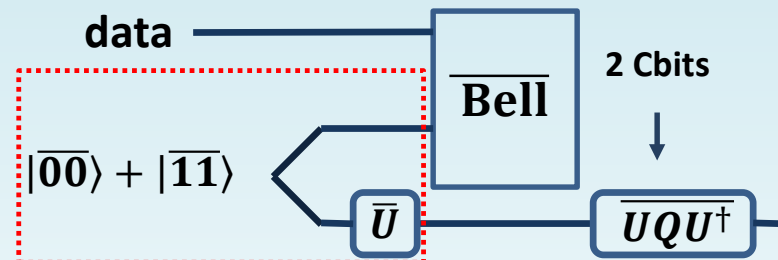
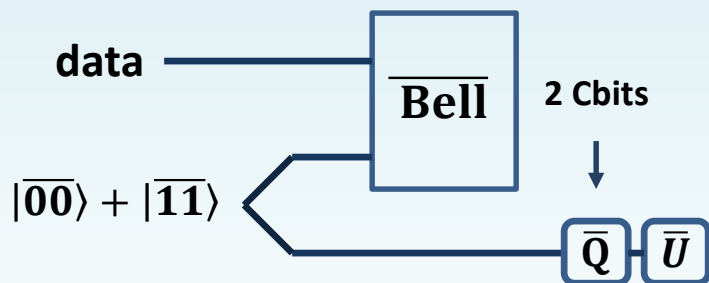
$$\mathbb{I} \otimes \bar{X} \rightarrow \mathbb{I} \otimes \bar{X}$$

$$\mathbb{I} \otimes \bar{Z} \rightarrow \bar{Z} \otimes \bar{Z}$$

Doubly-even self-dual CSS codes \rightarrow The best class of codes for FTQC

GATE TELEPORTATION AND UNIVERSAL FTQC

$$T_{\pi/8} : \begin{cases} X \rightarrow e^{i\pi/4} X P^\dagger \\ Z \rightarrow Z \end{cases}$$



$$(\mathbb{1} \otimes \bar{U})(|\bar{00}\rangle + |\bar{11}\rangle)$$

$$(\mathbb{1} \otimes \bar{T}_{\pi/8})(|\bar{00}\rangle + |\bar{11}\rangle)$$

More efficient gate teleportation

$$U_z(\theta) = \exp(-i\frac{\theta}{2}Z)$$

$$|A_\theta\rangle = U_z(\theta)|+\rangle$$

$$T_{\pi/8} = U_z(\pi/4)$$

