Wavelets in Quantum Computing

Mohsen Bagherimehrab
University of Toronto

PRX Quantum **3**, 020364 (2022)
Nearly optimal quantum algorithm for generating the ground state of a free quantum field theory

arXiv:2306.11802 (2023)
Fast quantum algorithm for differential equations

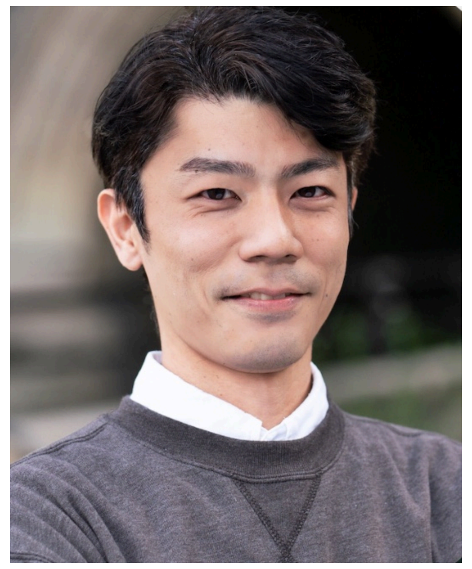
arXiv:2309.09350 (2023)
Efficient Quantum Algorithm for All Quantum Wavelet Transforms


Yuval Sanders


Dominic Berry


Gavin Brennen


Barry Sanders


Kouhei Nakaji


Nathan Wiebe


Alán Aspuru-Guzik

University of Tech. Sydney
Macquarie University
University Calgary
University of Toronto

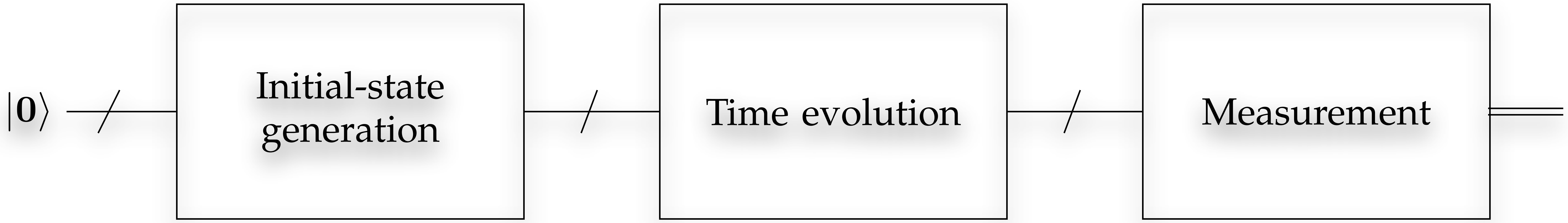
Key messages

- Wavelets are a key tool for analyzing the physics of continuum systems (e.g., quantum fields), especially for system with local defects.

- Knowing the physics of a quantum system, e.g. its symmetries, enables constructing superior quantum algorithms.

- Knowing a description of a quantum state vs generating the state on a quantum computer
  - efficient description is necessary but not sufficient

Simulating a QFT on a quantum computer



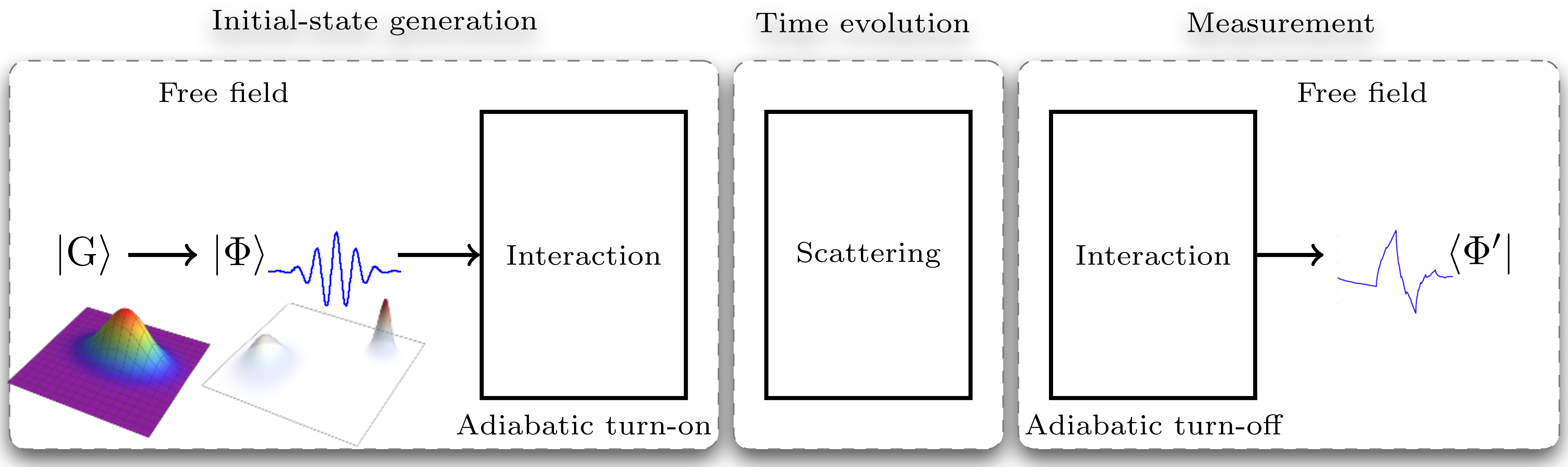
$|\mathbf{0}\rangle$

Initial-state generation

Time evolution

Measurement

Quantum simulation of scattering process [Jordan, Lee, Preskill `12 & `14]



Initial-state generation

Time evolution

Measurement

Free field

$|G\rangle \longrightarrow |\Phi\rangle$

Interaction

Adiabatic turn-on

Scattering

Interaction

Adiabatic turn-off

Free field

$\langle\Phi'|$

- Generating |G⟩ is a bottleneck for the entire simulation

Algorithmic quantum-state generation

Given:

- an *efficient* description of a quantum state $|\mathrm{G}\rangle$
- error tolerance ε

Task:

construct a quantum circuit $\mathscr{Q}: |\mathbf{0}\rangle \mapsto |\tilde{\mathrm{G}}\rangle$
such that $|\tilde{\mathrm{G}}\rangle$ is "ε-close" to $|\mathrm{G}\rangle$

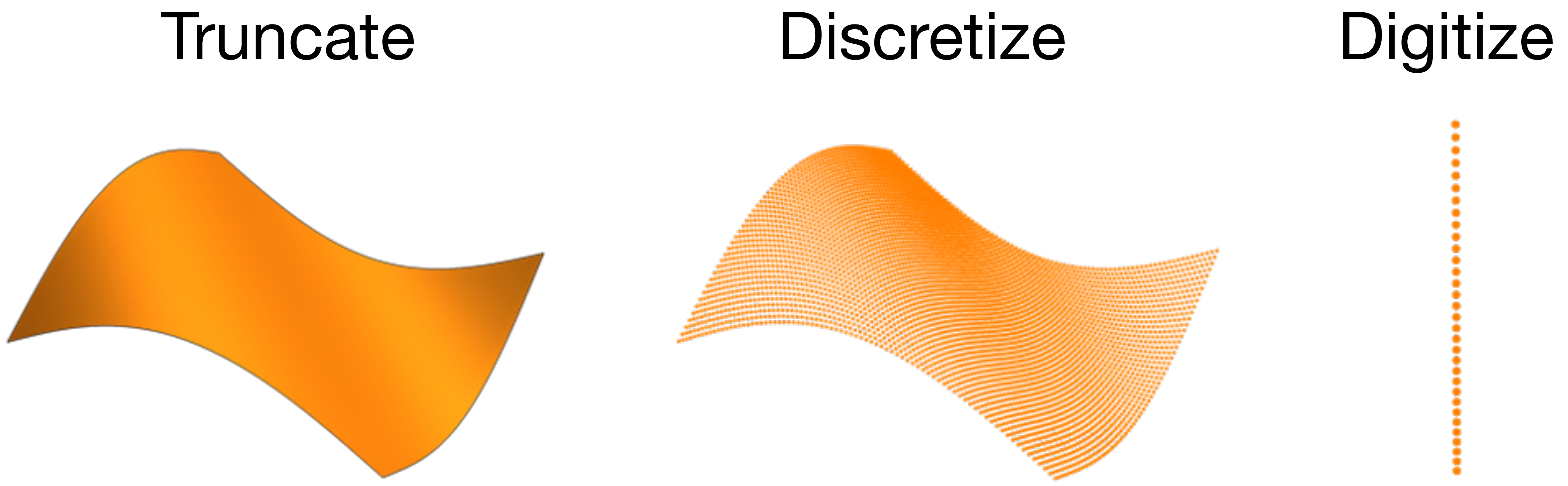
Massive scalar QFT with quartic interaction

$$\hat{H} = \hat{H}_{\text{free}} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{free}} = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \left[\hat{\Pi}^2(x) + \left(\nabla\hat{\Phi}(x)\right)^2 + m_0^2\hat{\Phi}^2(x)\right]$$

$$\hat{H}_{\text{int}} = \frac{\lambda_0}{4!}\int_{\mathbb{R}} \mathrm{d}x\, \hat{\Phi}^4(x)$$

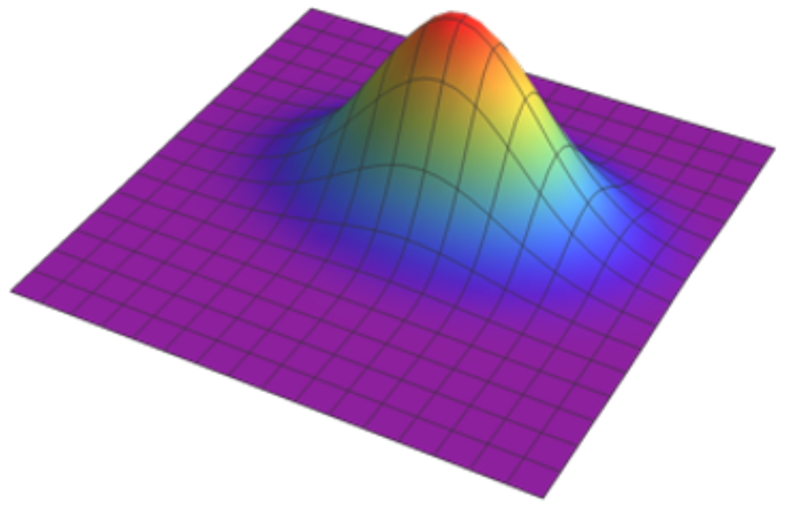
$$[\Phi(x), \Phi(y)] = \mathrm{i}\delta(x-y)\mathbb{I}$$



$$\hat{H}_{\text{disc}} = \frac{1}{2}\sum_{\ell}\hat{\Pi}_{\ell}^2 + \frac{1}{2}\sum_{\ell}\hat{\Phi}_{\ell}K_{\ell\ell'}\hat{\Phi}_{\ell'}, \qquad K \in \mathbb{R}^{N\times N}$$

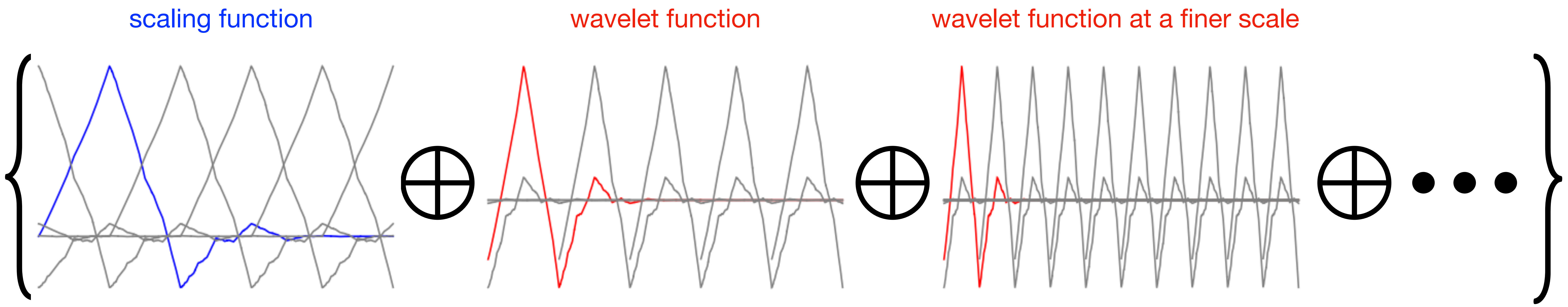
Ground State: a Gaussian state

$$|\mathrm{G}\rangle \propto \int_{\mathbb{R}^N} \mathrm{d}\phi\, \mathrm{e}^{-\frac{1}{4}\phi^{\intercal}\sqrt{K}\phi}\,|\phi\rangle$$



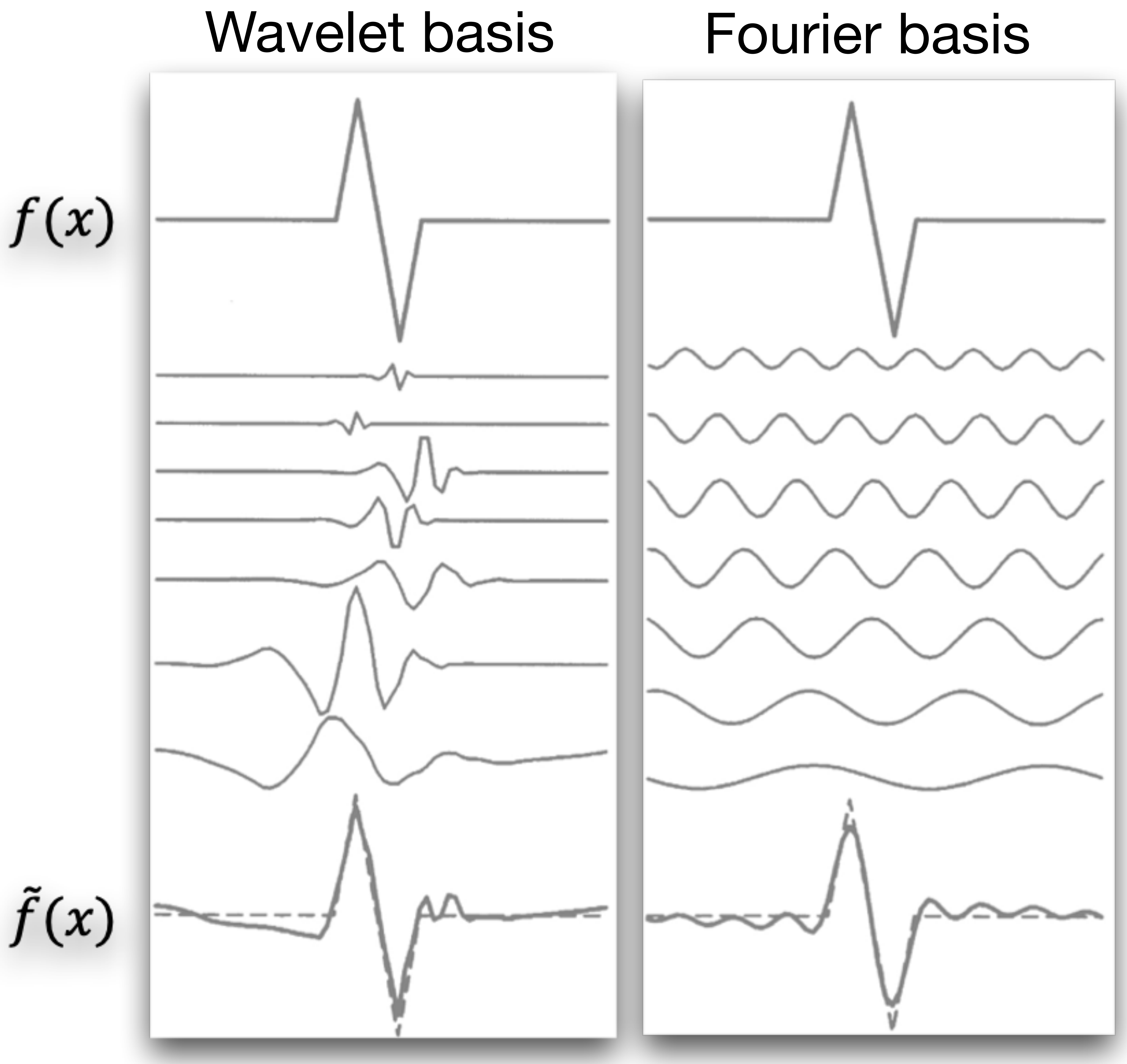
Wavelet basis

$$L^2(\mathbb{R}) = \mathrm{Span}\left\{ \cdots \oplus \cdots \oplus \cdots \oplus \cdots \right\}$$



- Locality in real and dual spaces
- Differentiability
- Sparse representations
  - Compression
  - Faster algorithms
- Efficient preconditioning

[Bagherimehrab, Nakaji, Wiebe, Aspuru-Guzik `23]



Wavelets zoo

Daubechies

Coiflet

Symlet

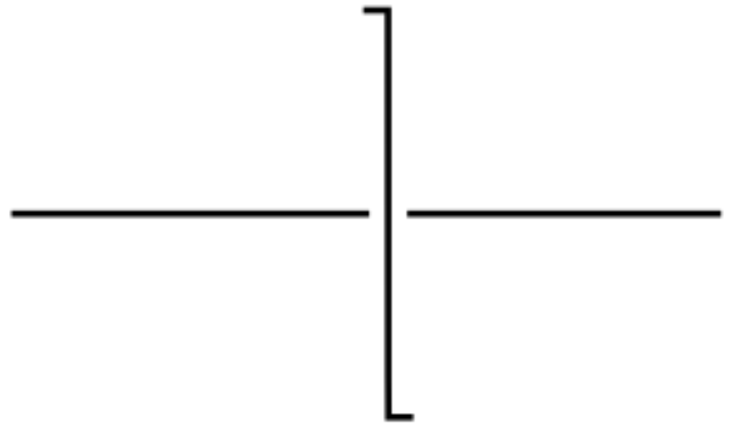
CDF 5/3

CDF 9/7

and many more!

Daubechies family

Index = 1



Index = 2

Index = 3

Index = 4

...

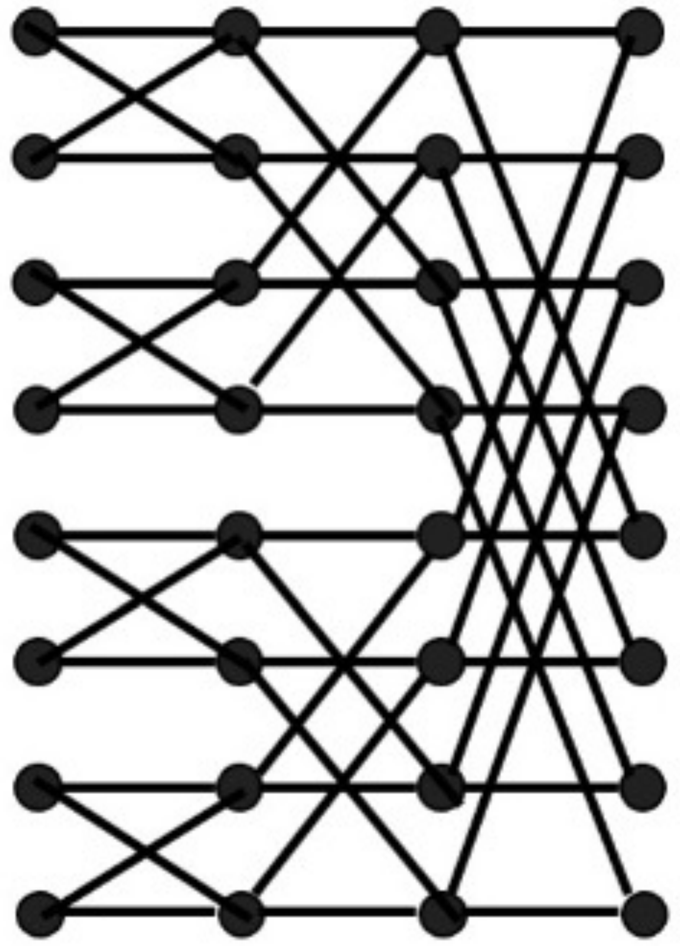
Index = 12

...

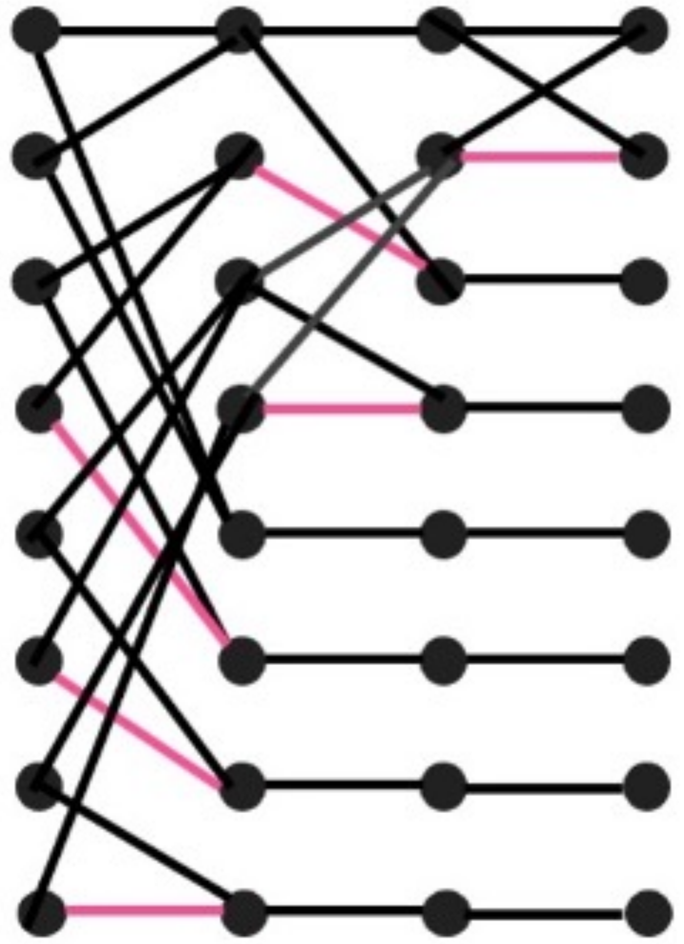
Specified by a sequence of numbers

Fourier vs wavelet transform

FFT

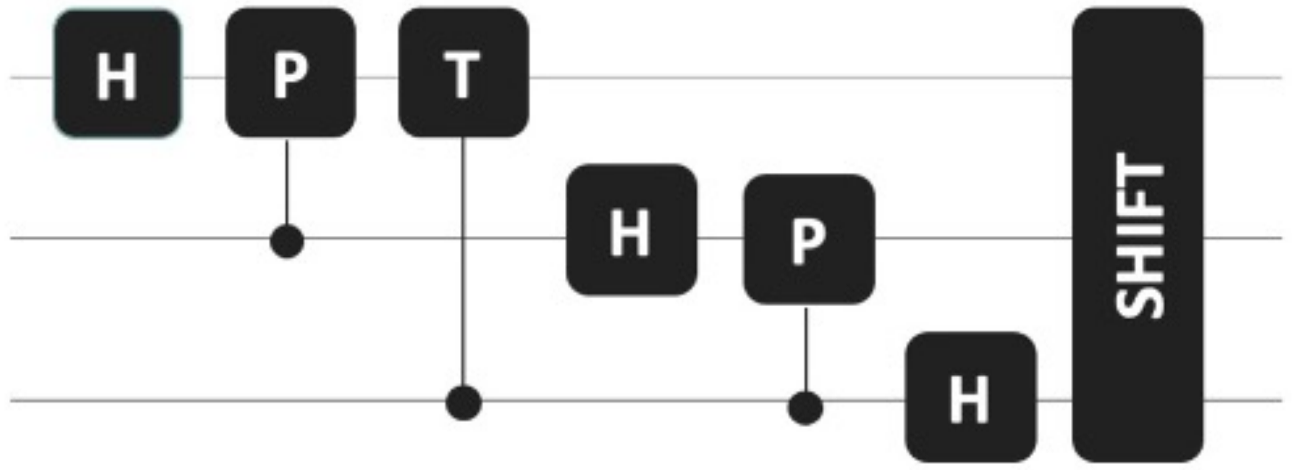


FWT

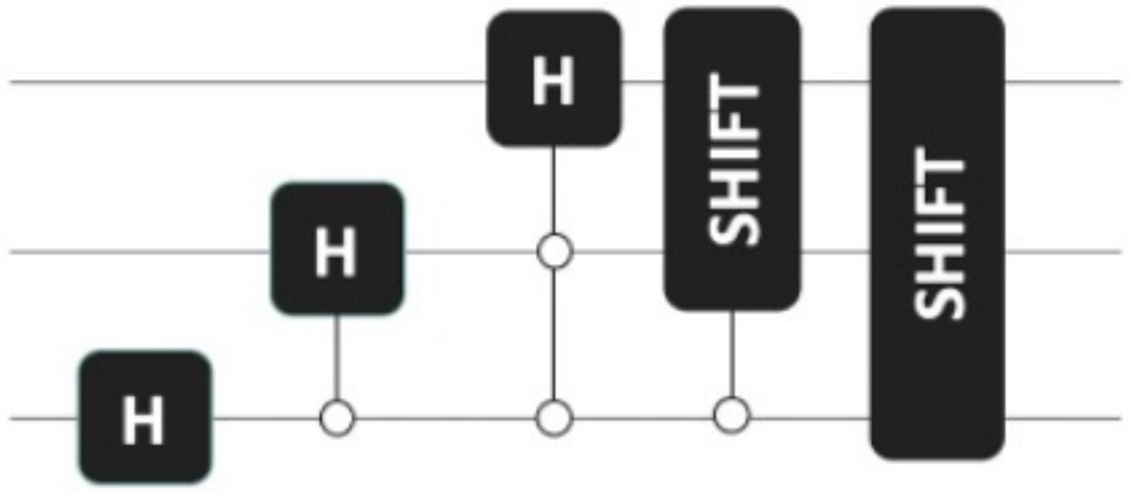


$$\mathcal{O}(N \log N)$$

QFT



QWT



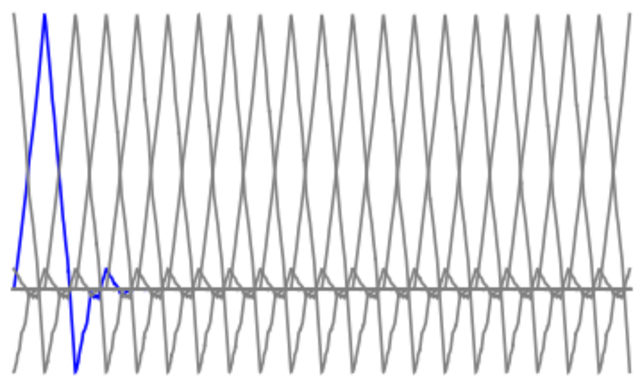
$$\mathcal{O}(n^2)$$

$$n = \log N$$

[Bagherimehrab, Aspuru-Guzik `23]

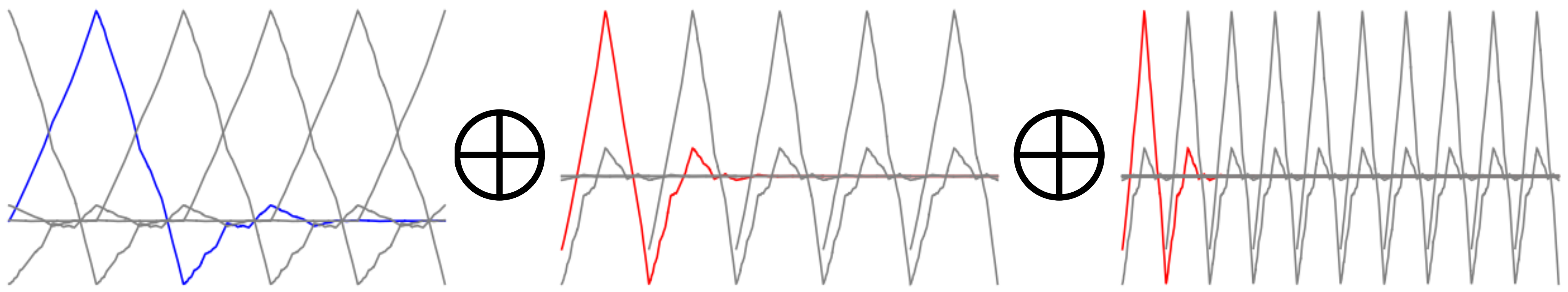
Discretization

Fixed-scale basis



$$L^2(\mathbb{R}) = \mathrm{Span}\left\{ \quad \cdots \right\}$$

Multi-scale wavelet basis



$$L^2(\mathbb{R}) = \mathrm{Span}\left\{ \quad \oplus \quad \oplus \quad \oplus \cdots \right\}$$

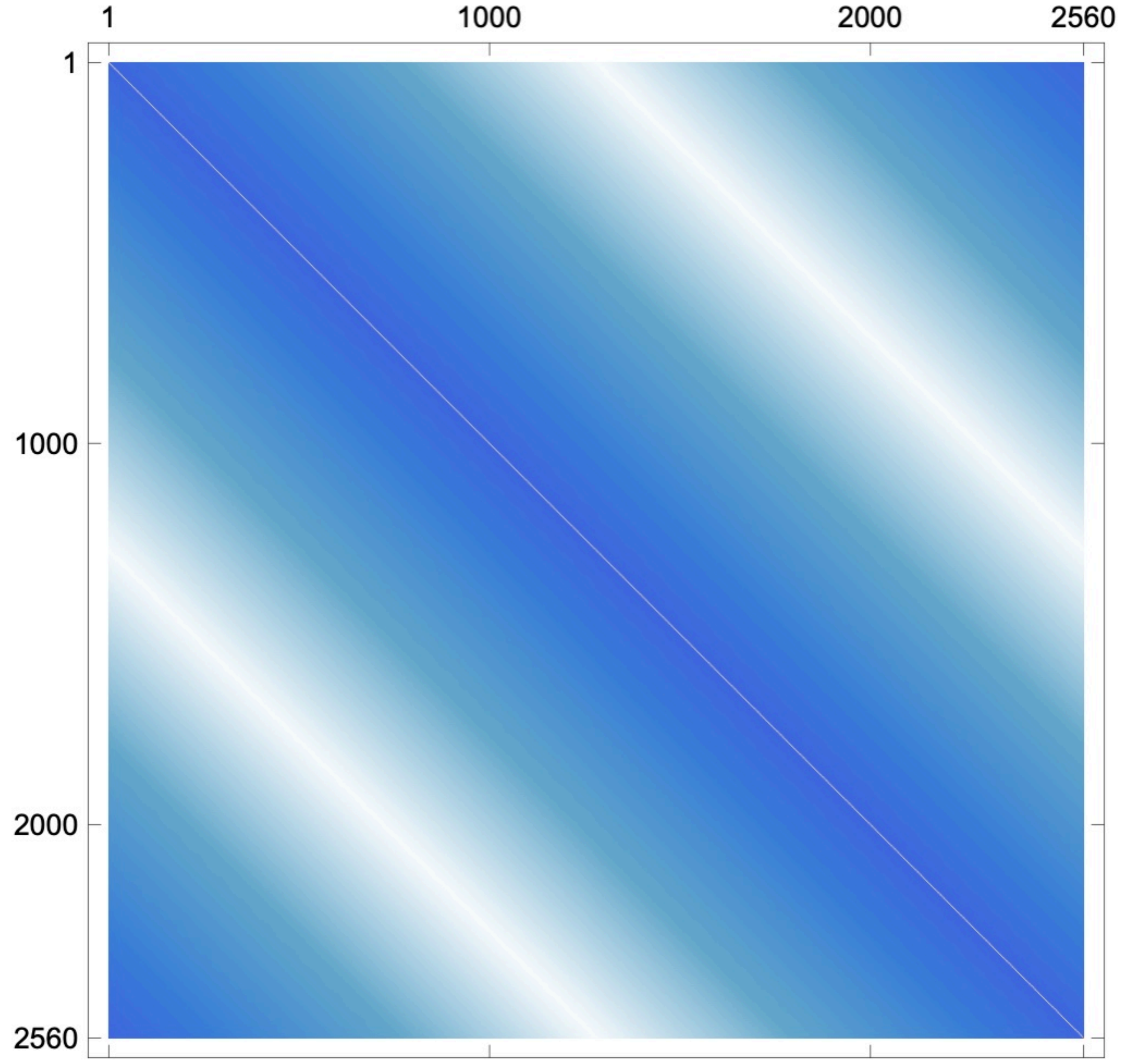
Discretized free QFT

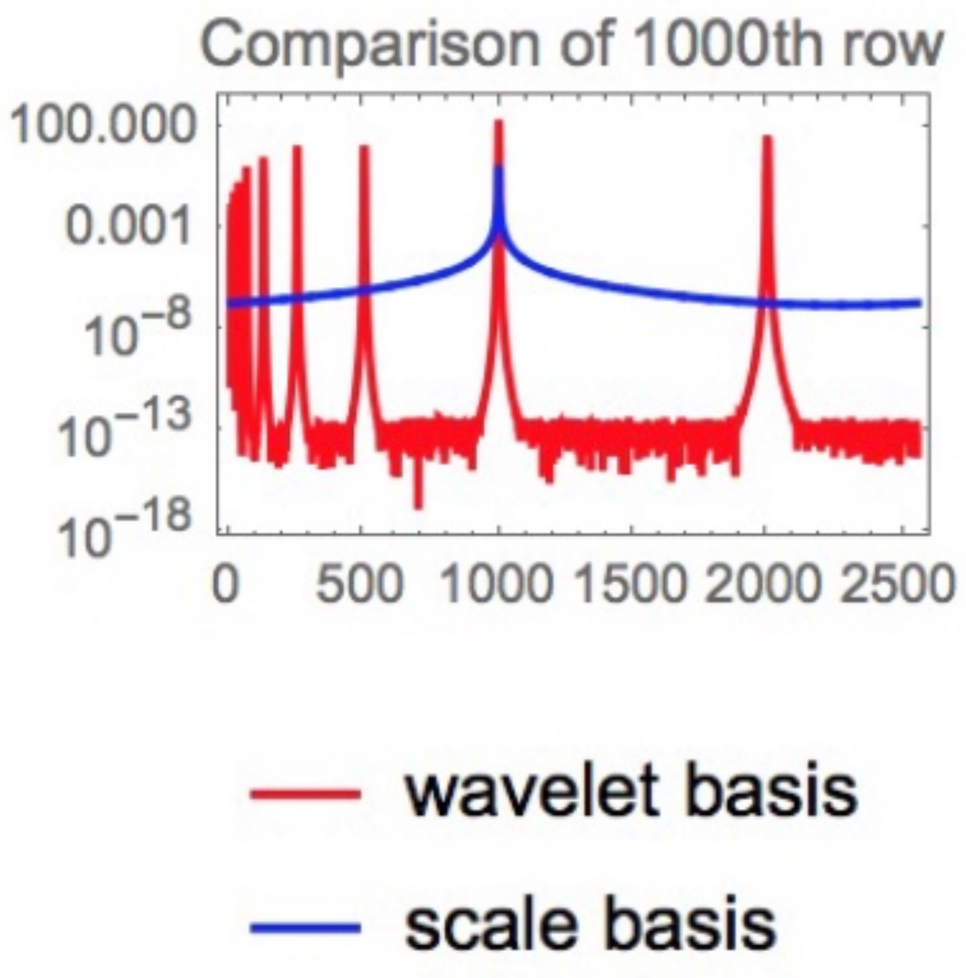
$$\hat{H}_{\mathrm{disc}} = \frac{1}{2}\sum_{\ell} \hat{\Pi}_{\ell}^{2} + \frac{1}{2}\sum_{\ell} \hat{\Phi}_{\ell} K_{\ell\ell'} \hat{\Phi}_{\ell'}, \quad K \in \mathbb{R}^{N\times N}$$

Ground State

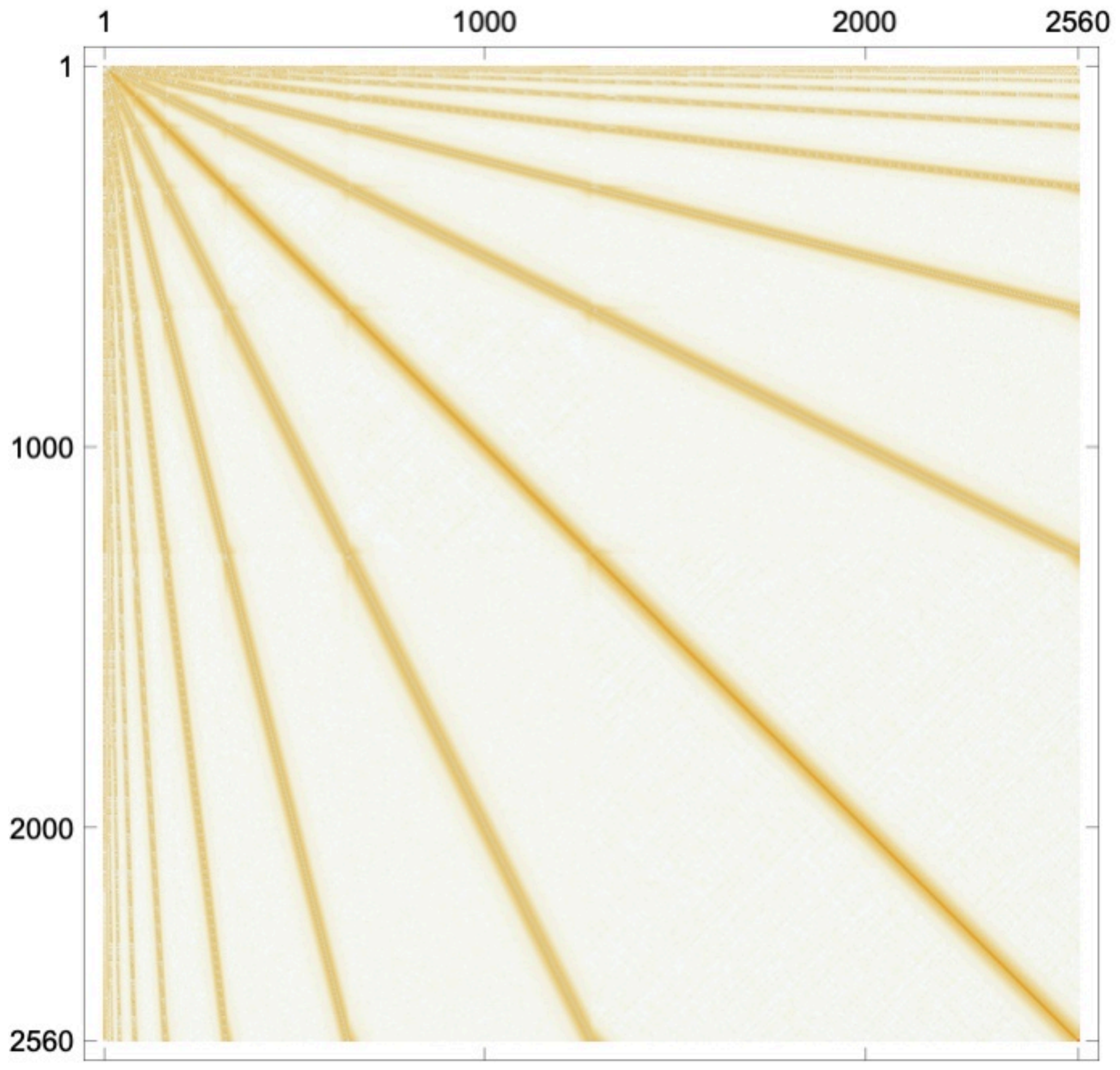
$$|\mathrm{G}\rangle \propto \int_{\mathbb{R}^N} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{4}\phi^{\mathsf{T}}\sqrt{K}\phi} \, |\phi\rangle$$

$\sqrt{K}$ in fixed-scale basis





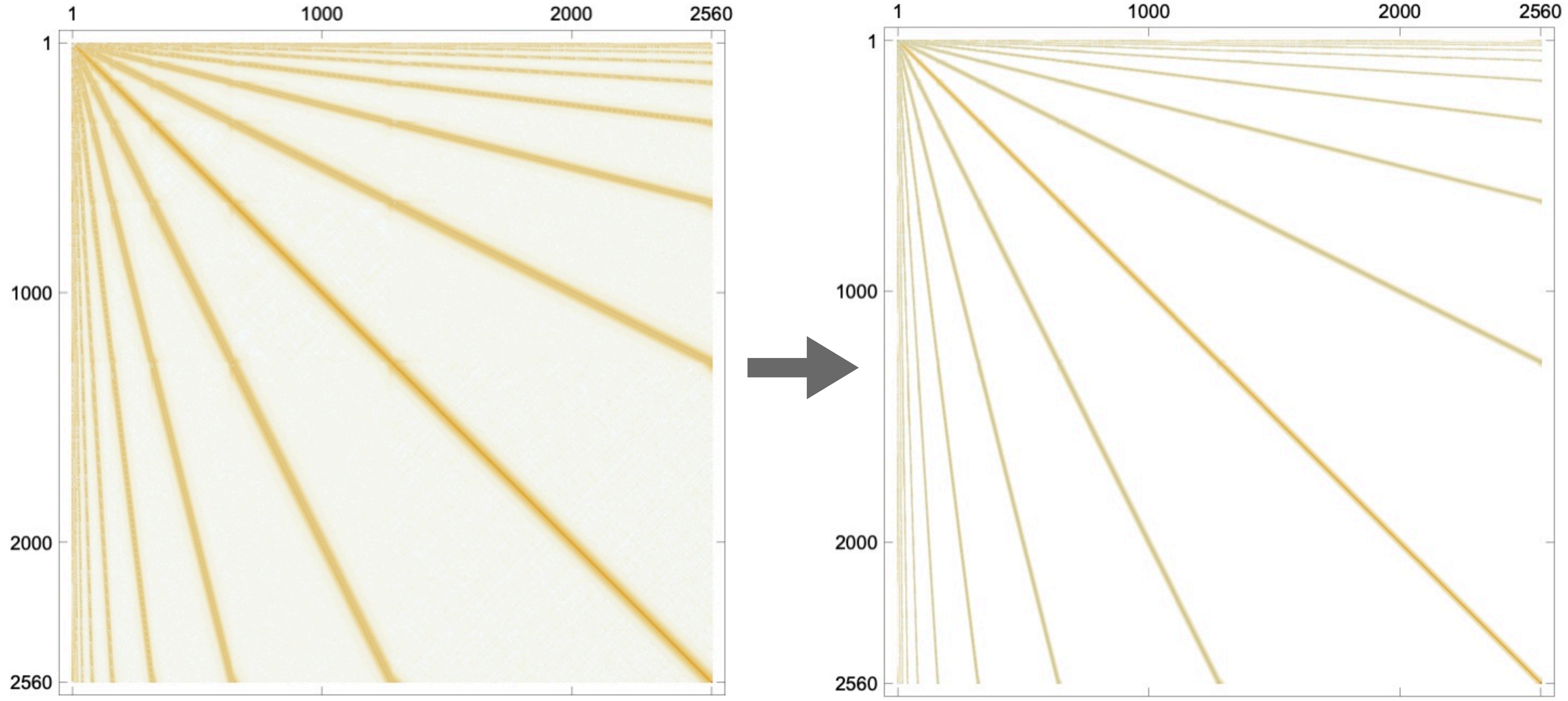
$\sqrt{K}$ in wavelet basis



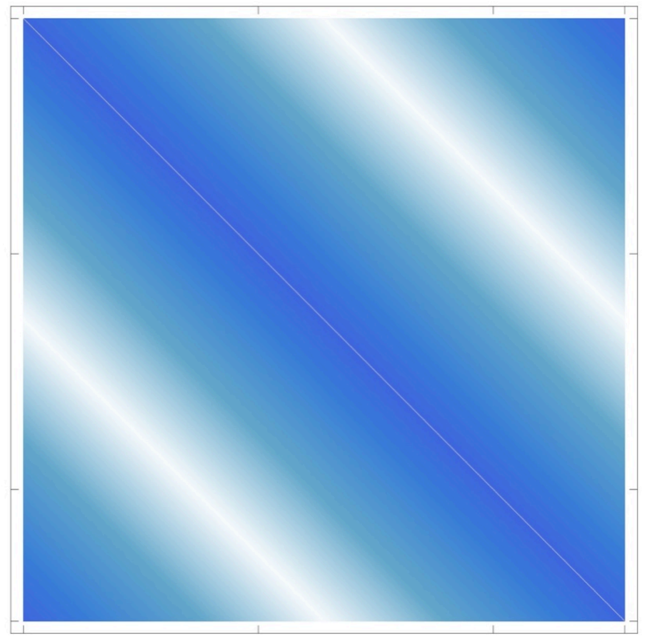
Ground-state generation

$\sqrt{K}$ in wavelet basis

truncated $\sqrt{K}$

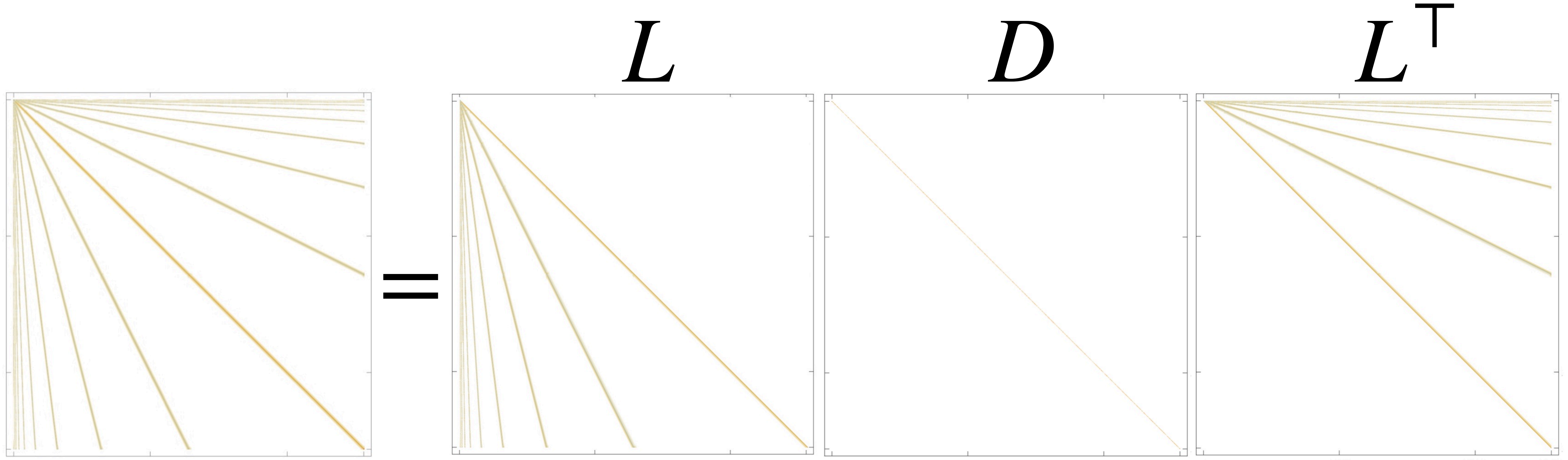


Finger-like sparse matrix



$= FDF^\dagger$

Circulant matrix



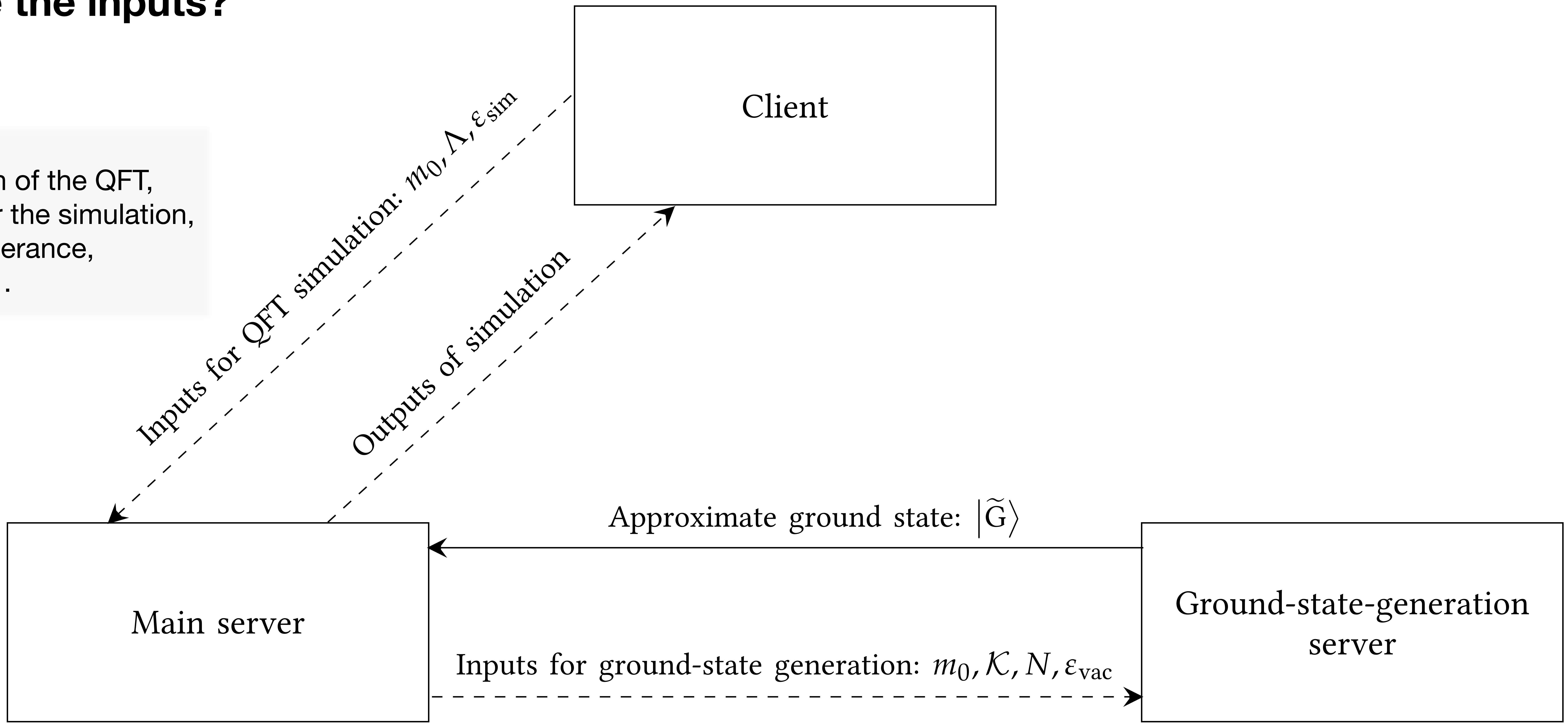
All diagonals are 1

A shear matrix

Client-server framework for quantum simulation of a QFT

What are the inputs?

Specification of the QFT,
energy scale for the simulation,
error tolerance,
…



Client

Inputs for QFT simulation: $m_0, \Lambda, \varepsilon_{\text{sim}}$

Outputs of simulation

Main server

Approximate ground state: $|\widetilde{\mathrm{G}}\rangle$

Ground-state-generation server

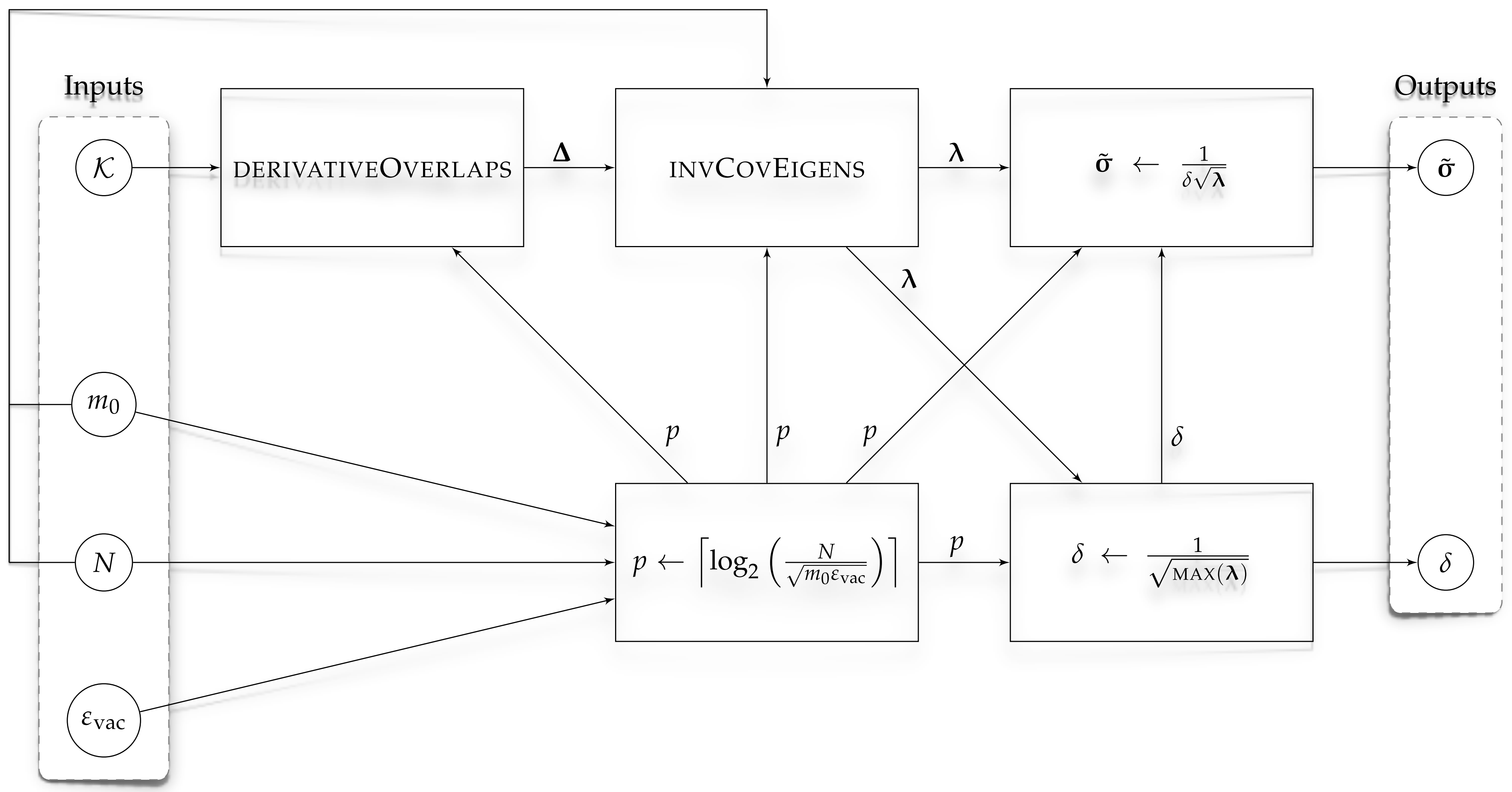
Inputs for ground-state generation: $m_0, \mathcal{K}, N, \varepsilon_{\text{vac}}$

Specification of the ground-state,
specification of basis,
error tolerance.

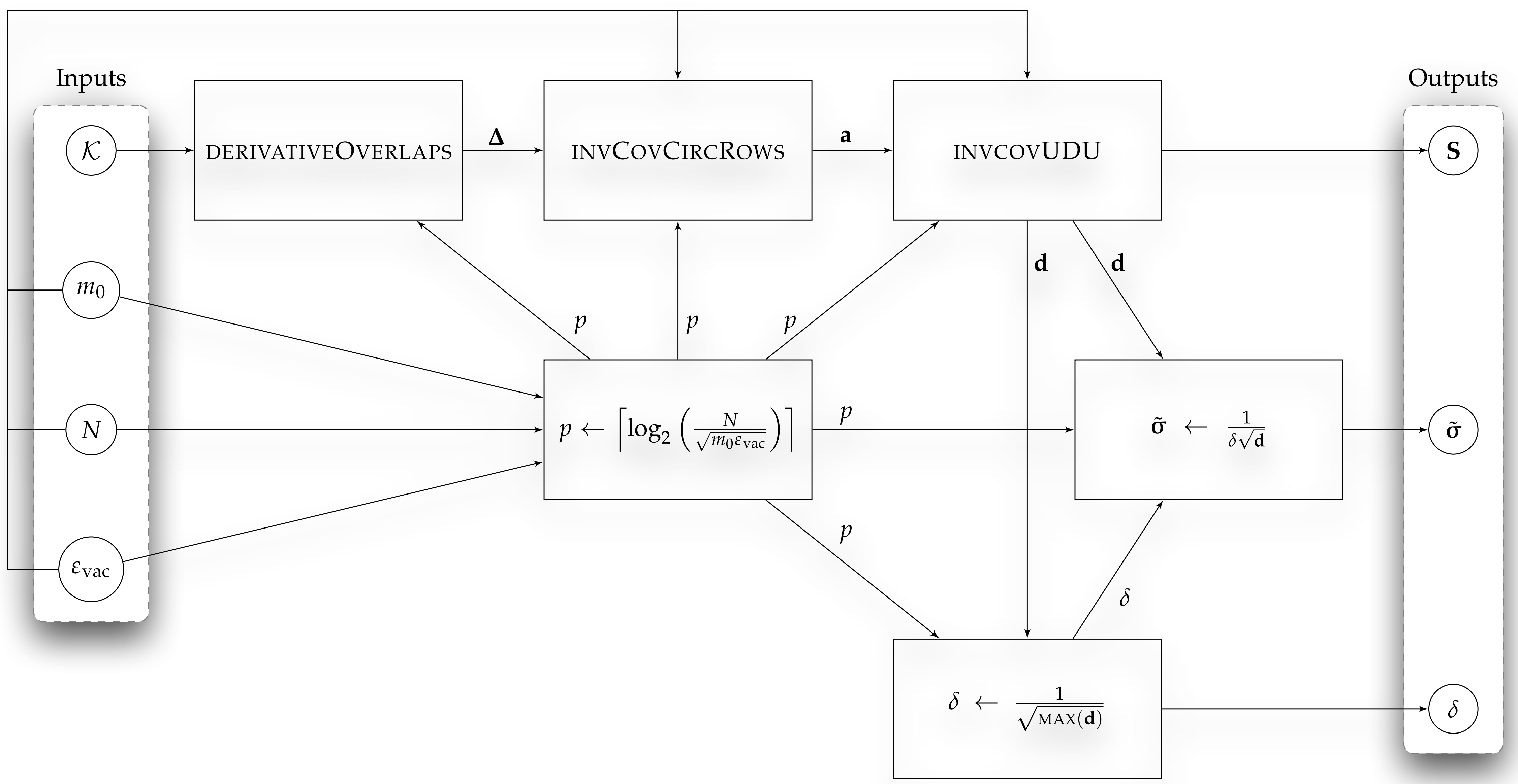
Ground-state generation

Classical

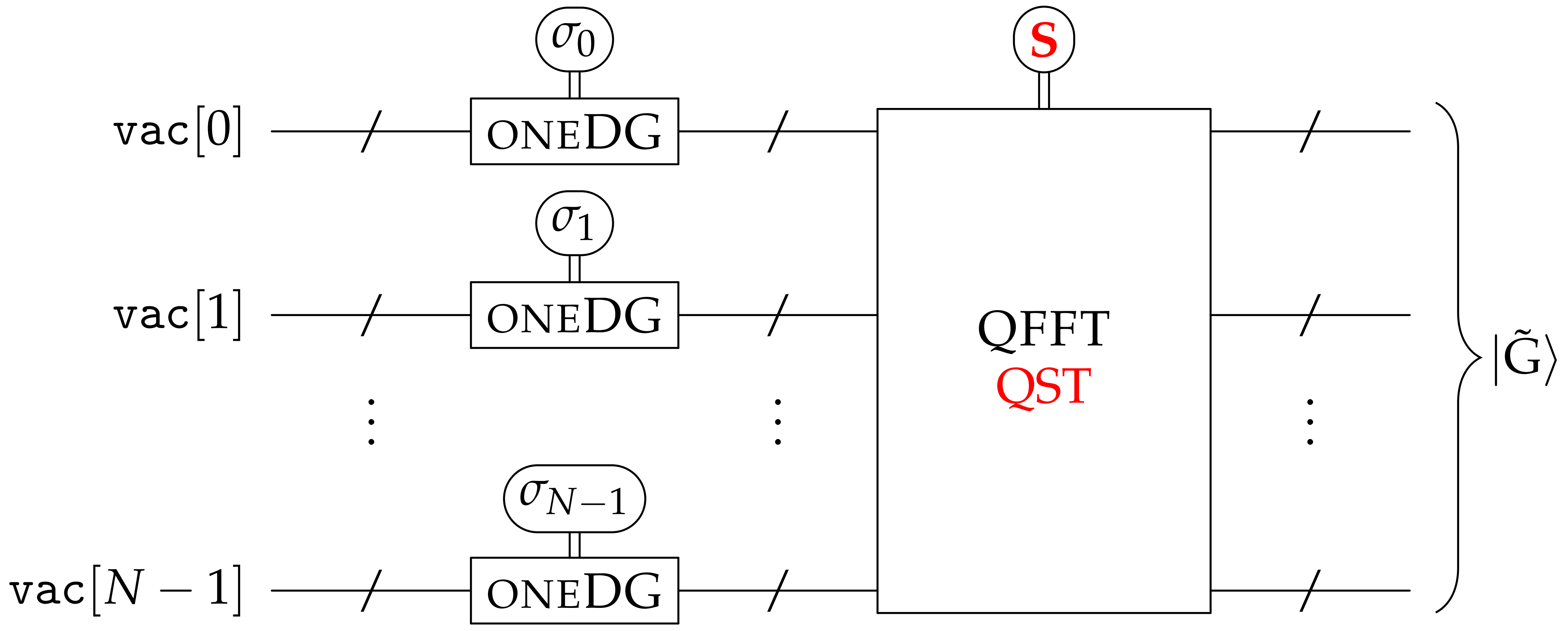
Fourier-based algorithm



Wavelet-based algorithm



Quantum



practical?

Cost: $\tilde{O}\left(N^{2.372}\right) \longrightarrow \tilde{O}(N)$

We show the cost is bounded by $\Omega(N)$.

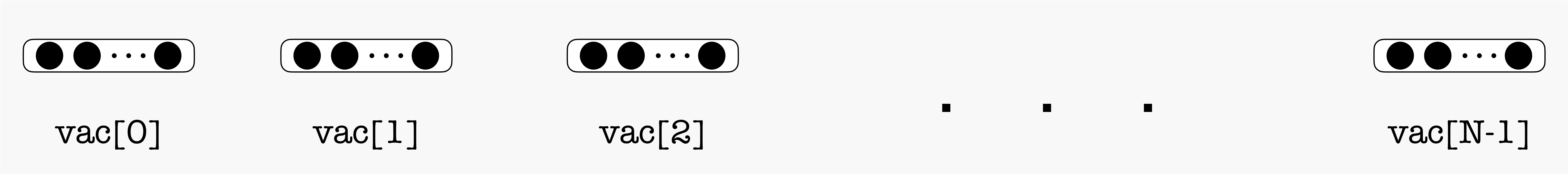
Our algorithms are optimal, up to log factors.

Proof sketch for the lower bound

Any sublinear algorithm will result in an exponentially bad approximation for the ground state.

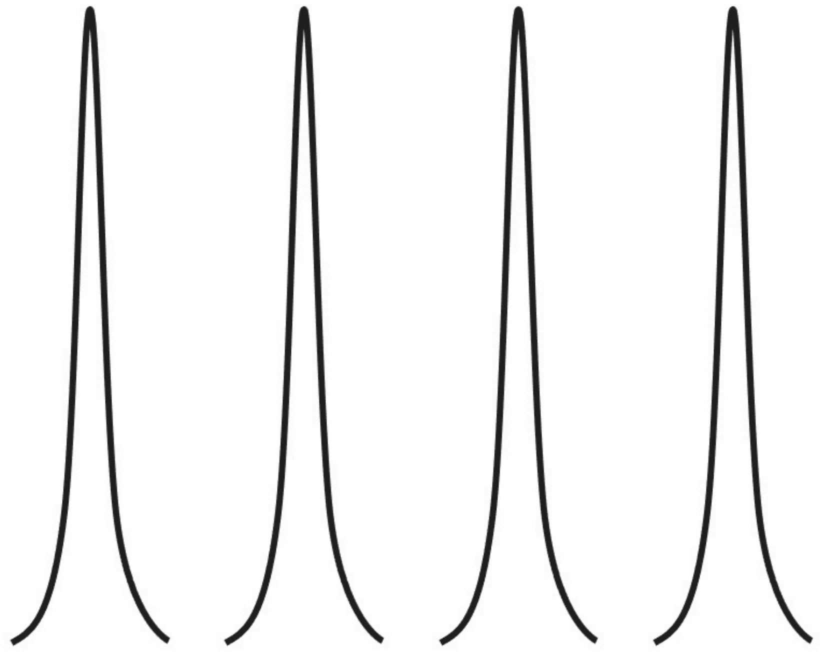
↓

N^α with $0 < \alpha < 1$



Infinite-mass theory (not physical theory): product of all-zero state.

Finite but large-mass theory: product of 1DGs with small variances.

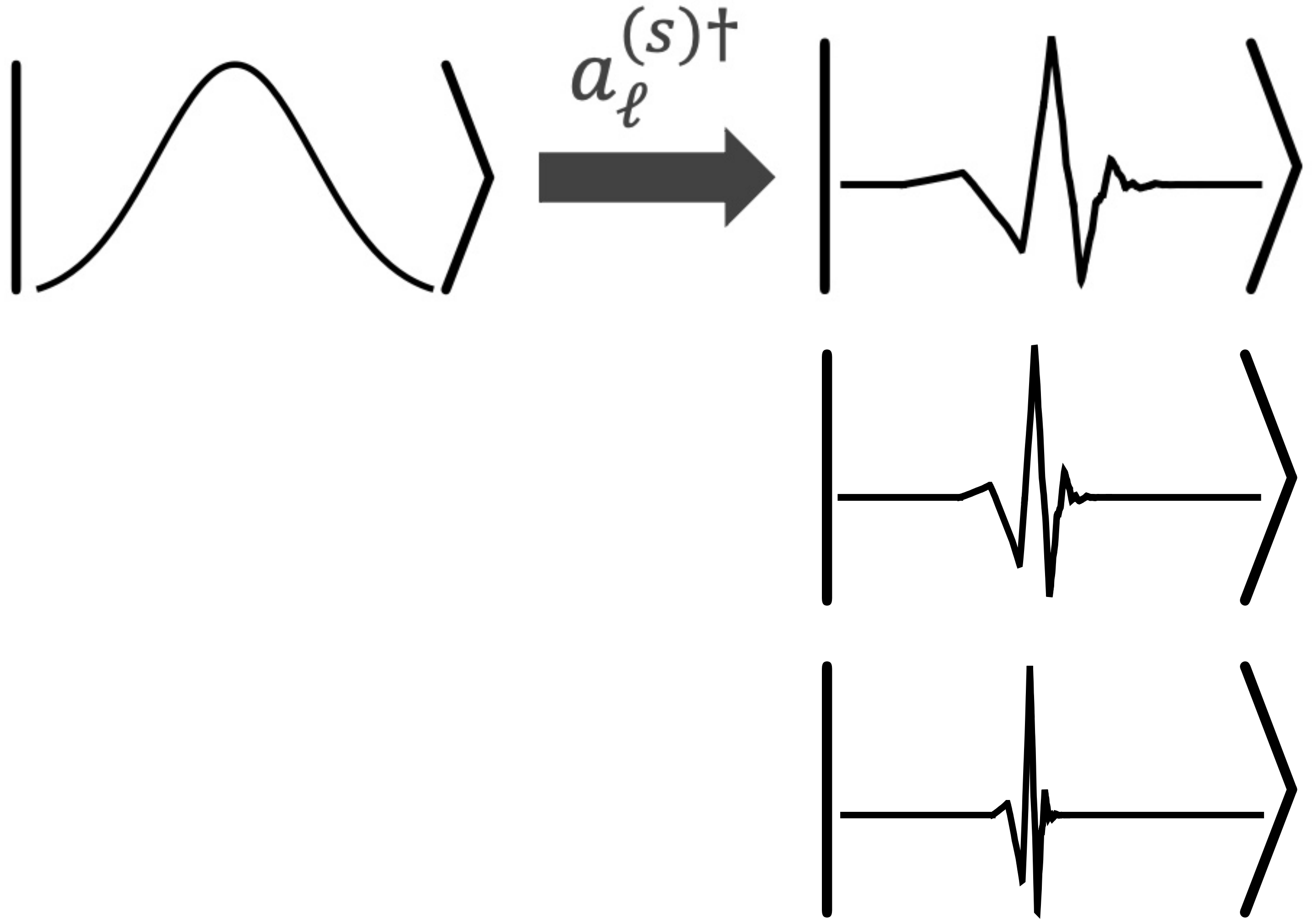


$$\langle G | \tilde{G} \rangle \le e^{-\frac{N-N^\alpha}{4m_0}}$$

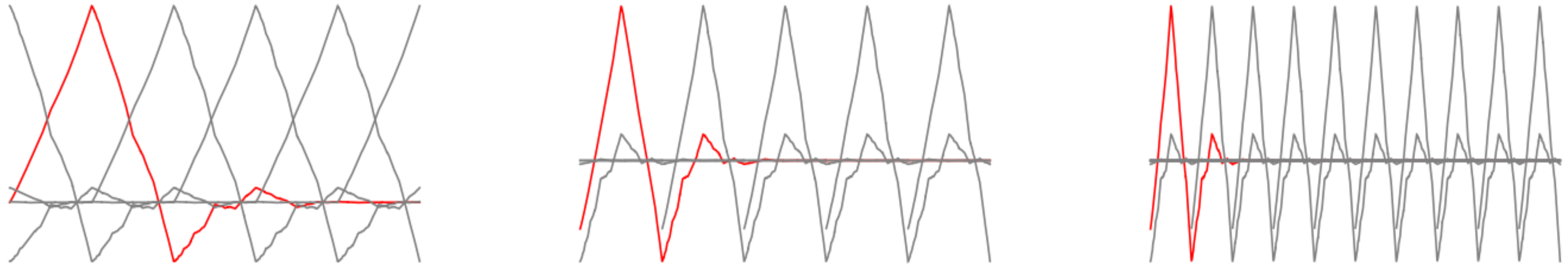
All other finite-mass theories: superposition of computational states with Gaussian amplitudes.

Why wavelet approach for simulating a QFT?

Preparing particle state at different energy scales



Provides description of QFT & all operators in multiple scales

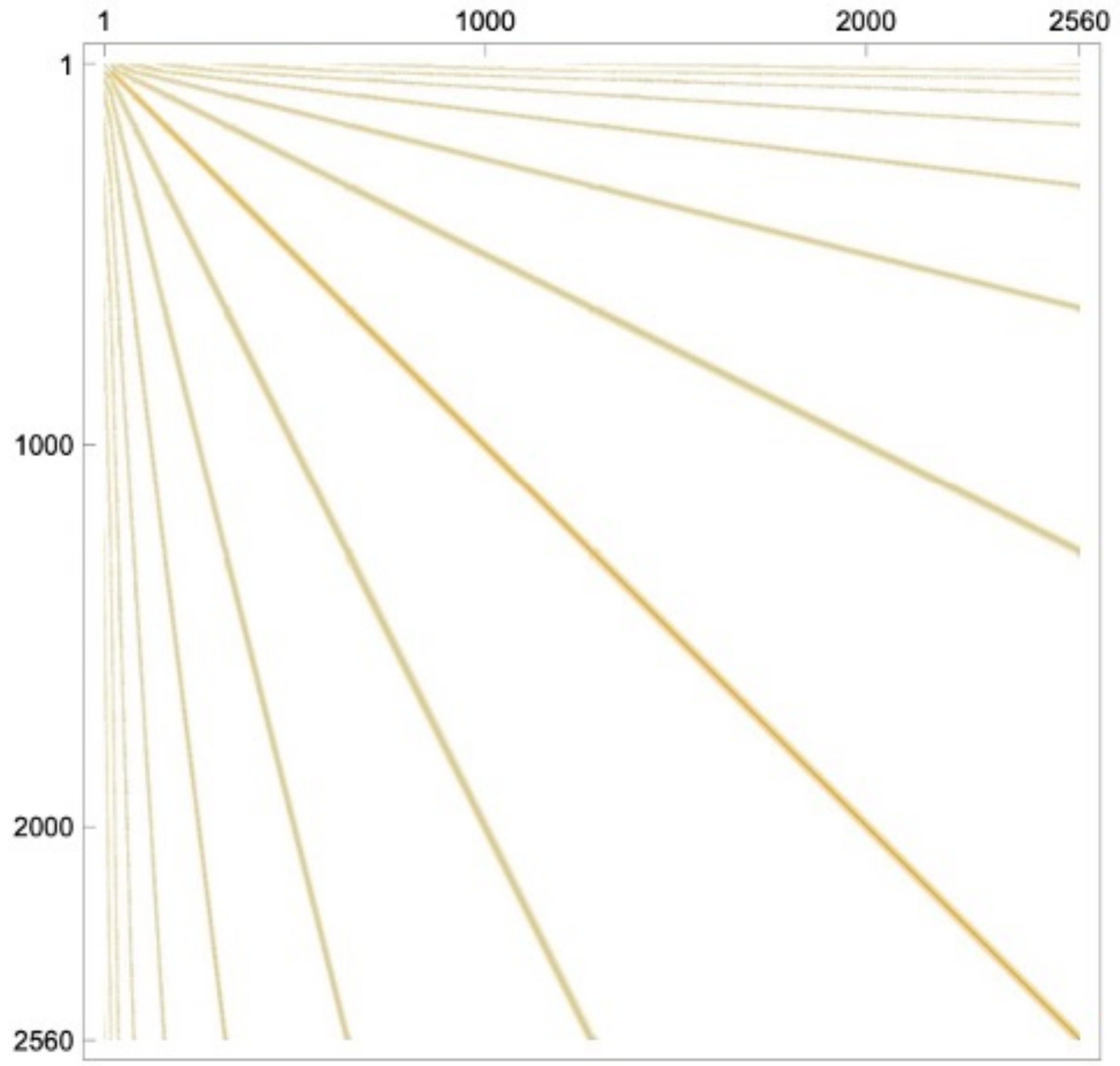


$$H \to H^{(s)}, \quad a^\dagger \to a^{(s)\dagger}$$

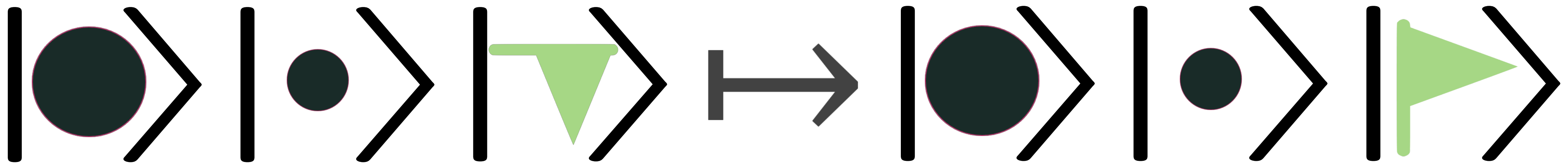
Beyond translationally invariant QFTs: **inhomogeneous-mass QFT**

Point defect: $m_0 \to m_0 + K\delta(x - x')$

For numerics: $K = 100m_0\,; m_0 \in [10^{-6}, 10^{6}]$



State preparation by inequality testing

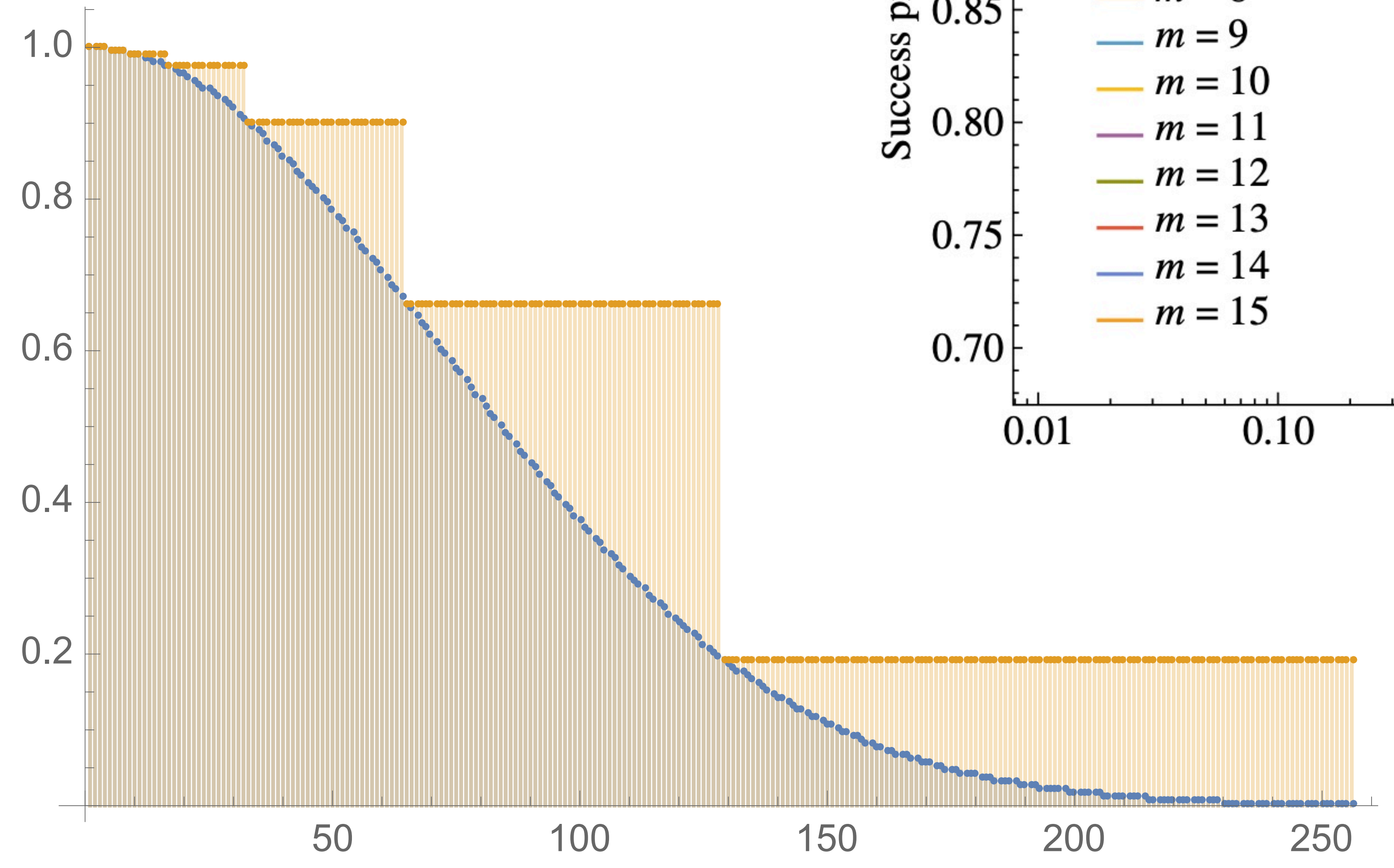


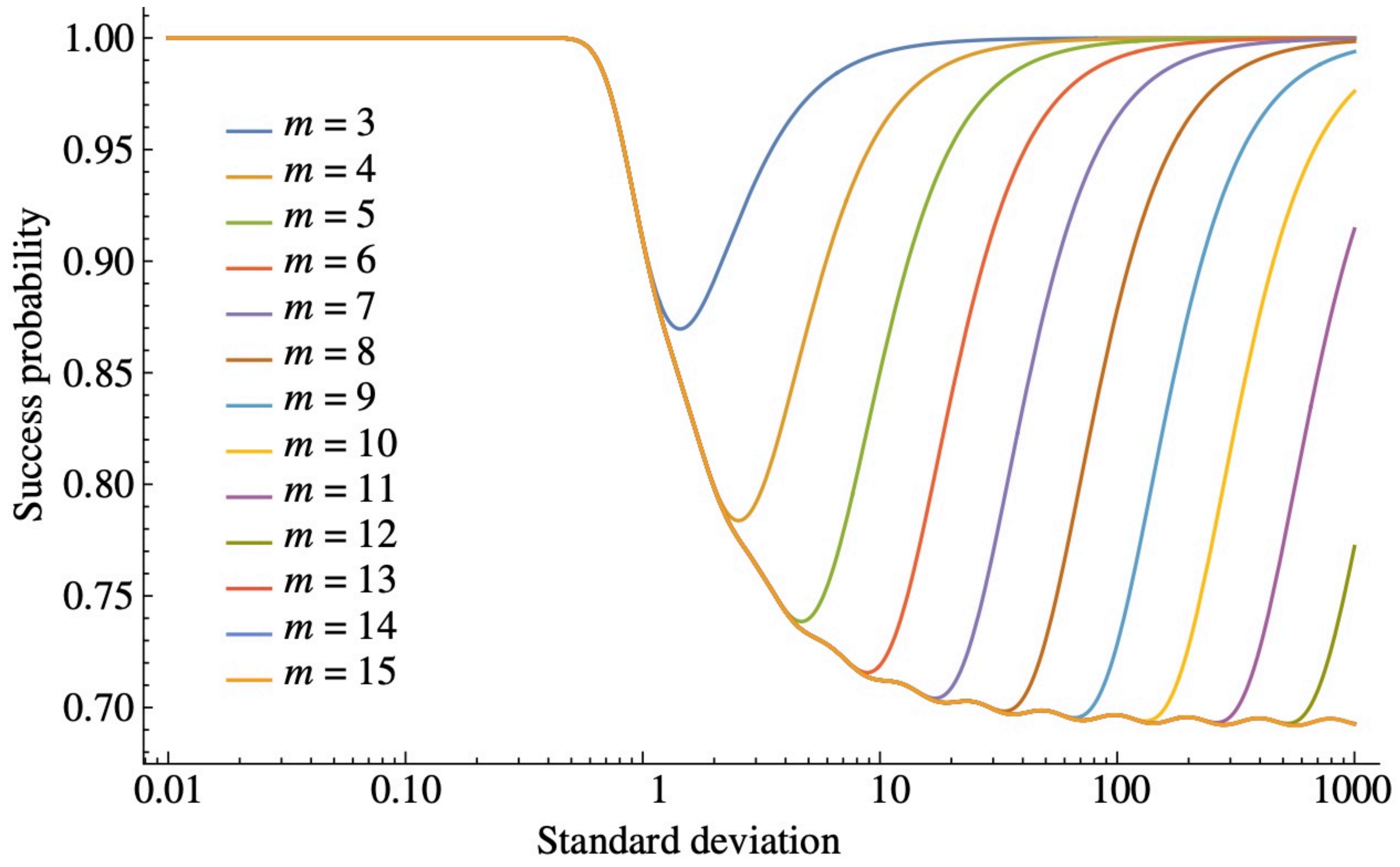
state to prepare: $|\psi\rangle = \sum_j f(j)\,|j\rangle_{\mathtt{out}}$

start with: $|\tilde{\psi}\rangle = \sum g(j)\,|j\rangle_{\mathtt{out}}$

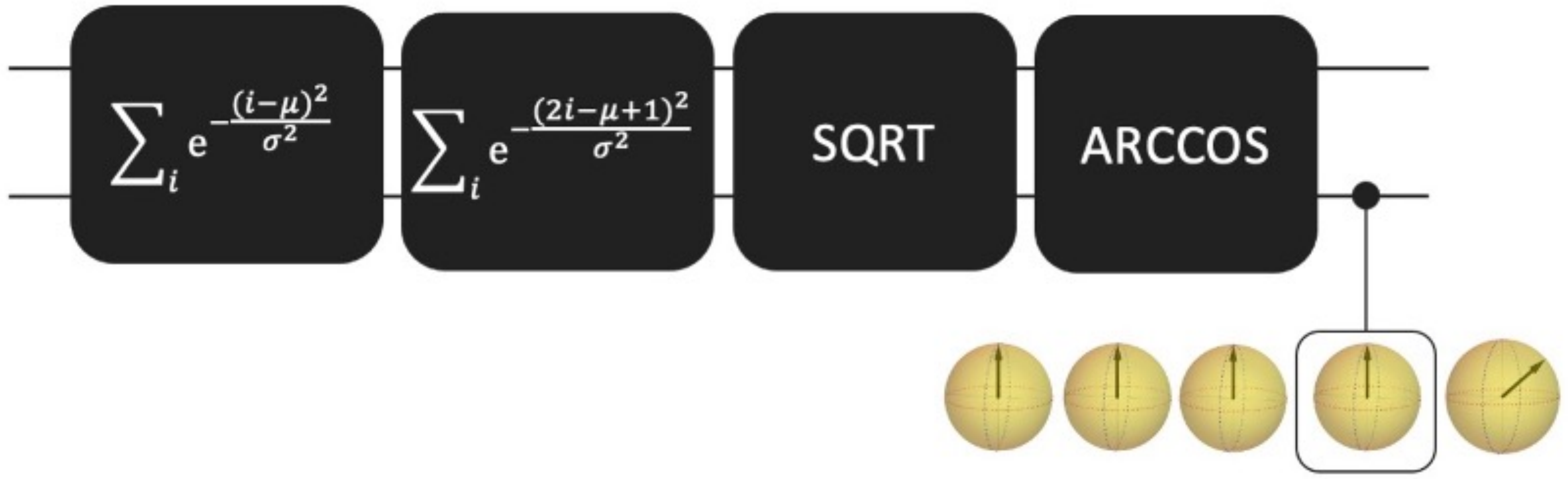
$$\sum_j g(j)\,|j\rangle_{\mathtt{out}} \mapsto \sum_j g(j)\,|j\rangle_{\mathtt{out}}\,|r(j)\rangle_{\mathtt{tmp}} \sum_z |z\rangle_{\mathtt{ref}}$$

$$r(j) = f(j)/g(j)$$





Standard method: requires costly quantum arithmetic [Kitaev, Webb `09]



Summary

- Wavelets are a key tool for analyzing the physics of quantum fields.
- Many physical system are structured (imposed by various symmetries) and structure enables designing superior quantum algorithms.
- Constructed two nearly optimal quantum algorithms for generating the ground state of the free massive QFT.
- Established a fundamental limit in simulating a bosonic QFT on a quantum computer.
- Inequality testing removes costly quantum arithmetic and could be useful for early-stage quantum computers.

Outlook

- Wavelet approach for measuring physical quantities in multiple scales (simulating a variable-length measurement apparatus in an experiment)

- Application of new Hamiltonian-simulation approaches for QFTs
  - Suzuki-Trotter approach has limitations (exponentially growing prefactors!)

- Does randomization help for QFT simulations?

- Hybridize approaches for QFT simulation